MATH 431 Written Homework 9 Solution Radford 04/09/2009

1. Page 415, number 36: (**30 points**) H is a normal subgroup of a finite group G and $|H| = p^\ell$ for some positive prime p and $\ell \geq 0$. We may assume that $\ell > 0$. By Sylow's Second Theorem $H \subseteq K$ for some Sylow p-subgroup K of G (**10**). Any Sylow p-subgroup of G has the form gKg^{-1} for some $g \in G$ by Sylow's Third Theorem (**10**). Thus $H = gHg^{-1} \subseteq gKg^{-1}$ since H is normal and $H \subseteq K$ (**10**). We have shown that H is contained in every Sylow p-subgroup of G.

2. Page 415, number 40: (**30 points**) If $|G| = 1$ then $|G| = p^0$ is a power of p. Suppose $|G| > 1$ and q is a positive prime divisor of $|G|$ (**7**). Then G has an element a of order q by Cauchy's Theorem (**7**). But the order of a is p^ℓ for some $\ell \geq 0$ by assumption. Therefore $q = p^\ell$ which implies $q = p$ (**8**). Since p is the only positive prime which divides $|G|$ it follows that $|G|$ is a power of p (**8**).

3. Page 415, number 44: (**40 points**) Suppose $|G| = 45 = 3^2 \cdot 5$. Let H be a Sylow 3-subgroup of G. Then $|H| = 3^2$ and is thus abelian by the corollary to Theorem 24.4. Let K be a Sylow 5-subgroup of G. Then $|H| = 5$ and is thus abelian since it is cyclic (**6**).

Let n_p be the number of Sylow p-subgroups of G for $p = 3, 5$. Since $n_5 | 3^2$ and $n_5 = 1 + 5\ell$ for some $\ell \geq 0$ it follows that $n_5 = 1$. Likewise $n_3 | 5$ and $n_3 = 1 + 3\ell$ for some $\ell \geq 0$ which implies $n_3 = 1$. Thus H, K are normal subgroups of G by the corollary to Theorem 24.5 (**6**)

Now $H \cap K \subseteq H, K$ implies $|H \cap K|$ divides $|H| = 9$ and $|K| = 5$ by Lagrange's Theorem. Therefore $|H \cap K| = 1$ which means $H \cap K = (e)$. Since $|HK| = |H||K|/|H \cap K| = |H||K| = |G|$ it follows that $HK = G$ (**6**). Since H, K are normal and $H \cap K = (e)$ recall that $hk = kh$ by □□ on page 411 of the text (**6**).

We show that G is abelian. Let $g, g' \in G$. Then $g = hk$ and $g' = h'k'$ for some $h, h' \in H$ and $k, k' \in G$. Therefore

$$gg' = hkh'k' = h\underline{kh'}k' = hh'kk' = \underline{hh'}\,\underline{kk'} = h'hk'k = h'\underline{hk'}k = h'k'hk = g'g$$

which shows that G is abelian (**6**).

Since G is abelian $G \simeq \mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_5$ (**5**) or $G \simeq \mathbf{Z}_{3^2} \times \mathbf{Z}_5$ (**5**) by the Fundamental Theorem for Finite Abelian Groups.