1. Page 415, number 36: (30 points) \( H \) is a normal subgroup of a finite group \( G \) and \( |H| = p^\ell \) for some positive prime \( p \) and \( \ell \geq 0 \). We may assume that \( \ell > 0 \). By Sylow’s Second Theorem \( H \subseteq K \) for some Sylow \( p \)-subgroup \( K \) of \( G \) (10). Any Sylow \( p \)-subgroup of \( G \) has the form \( gKg^{-1} \) for some \( g \in G \) by Sylow’s Third Theorem (10). Thus \( H = gHg^{-1} \subseteq gKg^{-1} \) since \( H \) is normal and \( H \subseteq K \) (10). We have shown that \( H \) is contained in every Sylow \( p \)-subgroup of \( G \).

2. Page 415, number 40: (30 points) If \( |G| = 1 \) then \( |G| = p^0 \) is a power of \( p \). Suppose \( |G| > 1 \) and \( q \) is a positive prime divisor of \( |G| \) (7). Then \( G \) has an element \( a \) of order \( q \) by Cauchy’s Theorem (7). But the order of \( a \) is \( p^\ell \) for some \( \ell \geq 0 \) by assumption. Therefore \( q = p^\ell \) which implies \( q = p \) (8). Since \( p \) is the only positive prime which divides \( |G| \) it follows that \( |G| \) is a power of \( p \) (8).

3. Page 415, number 44: (40 points) Suppose \( |G| = 45 = 3^2 \cdot 5 \). Let \( H \) be a Sylow 3-subgroup of \( G \). Then \( |H| = 3^2 \) and is thus abelian by the corollary to Theorem 24.4. Let \( K \) be a Sylow 5-subgroup of \( G \). Then \( |K| = 5 \) and is thus abelian since it is cyclic (6).

Let \( n_p \) be the number of Sylow \( p \)-subgroups of \( G \) for \( p = 3, 5 \). Since \( n_5|3^2 \) and \( n_3 = 1 + 5\ell \) for some \( \ell \geq 0 \) it follows that \( n_5 = 1 \). Likewise \( n_3|5 \) and \( n_3 = 1 + 3\ell \) for some \( \ell \geq 0 \) which implies \( n_3 = 1 \). Thus \( H, K \) are normal subgroups of \( G \) by the corollary to Theorem 24.5 (6).

Now \( H \cap K \subseteq H, K \) implies \( |H \cap K| \) divides \( |H| = 9 \) and \( |K| = 5 \) by Lagrange’s Theorem. Therefore \( |H \cap K| = 1 \) which means \( H \cap K = (e) \). Since \( |HK| = |H||K|/|H \cap K| = |H||K| = |G| \) it follows that \( HK = G \) (6). Since \( H, K \) are normal and \( H \cap K = (e) \) recall that \( hk = kh \) by \( \square \) on page 411 of the text (6).

We show that \( G \) is abelian. Let \( g, g' \in G \). Then \( g = hk \) and \( g' = h'k' \) for some \( h, h' \in H \) and \( k, k' \in G \). Therefore

\[
gg' = hh'k'k = hh'kk' = hh'k'k = h'hk'k = h'hk'hk = g'g
\]

which shows that \( G \) is abelian (6).

Since \( G \) is abelian \( G \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \) (5) or \( G \simeq \mathbb{Z}_{3^2} \times \mathbb{Z}_5 \) (5) by the Fundamental Theorem for Finite Abelian Groups.