MATH 431 # Written Homework 10 Solution **Radford**

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1. Page 449, number 8: (**20 points**) Let $G = <\alpha, \beta>$, where $\alpha = (1\,2)(3\,4)$ and $\beta = (2\,4)$. Then $G = <\alpha\beta, \beta>$ and $b = \alpha\beta = (1\,2)(3\,4)(2\,4) = (1\,2\,3\,4)$. Set $a = \beta$. Then $\boxed{a^2 = b^4 = \mathrm{I}}$; indeed the order of a is 2 and the order of b is 4. (**6**)

The calculation

$$aba^{-1} = aba = (2\,4)(1\,2\,3\,4)(2\,4) = (1\,4\,3\,2) = (1\,2\,3\,4)^{-1} = b^{-1}$$

shows that $\boxed{ab = b^3a}$. Using the last boxed relation one can show that the elements of G, that is the expressions of the form $a^{m_1}b^{n_1}\cdots a^{m_r}b^{n_r}$, where $r \geq 1$ and $m_i, n_i \geq 0$ can be written as $b^m a^n$ for some $n, m \geq 0$ and hence $b^m a^n$, where $0 \leq m < 4$ and $0 \leq n < 2$ by the first boxed relations. Therefore $|G| \leq 4{\cdot}2 = 8$. (**7**)

Now $|<b>| = 4$ so 4 divides $|G|$ by Lagrange's Theorem. Hence $|G| = 4s$ for some $s \geq 1$. Since $|G| \leq 8$ either $s = 1$ or $s = 2$. Note $a \notin <b>$ as a is an odd permutation and cyclic group $<b>$ consists of even permutations. Therefore $|G| > 4$ which means $s = 2$ and $|G| = 8$. We have shown that $G \simeq D_4$. (**7**)

2. Page 449, number 12(a): (**20 points**) $a^2 = b^4 = e$, $ab = b^3a$. Therefore $aba = b^3$ and

$$a^3b^2abab^3 = a^3b^2\underline{aba}b^3 = a^3b^2\underline{b^3}\,b^3 = \underline{a^3}\,\underline{b^8} = \underline{a}\,\underline{e} = a.$$

3. Page 450, number 16: (**40 points**) Suppose $G \neq (e)$ and $|G| \leq 11$. Then $|G| = 8$, in which case G is classified by Theorem 26.4 (**5**), or for some positive prime p the order of G is (1) p (**5**), (2) p^2, or (3) $2p$ where $2 < p$. If (1) holds then $G \simeq \mathbf{Z}_p$. If (2) holds then G is abelian by the corollary to Theorem 24.2. Thus $G \simeq \mathbf{Z}_{p^2}$ or $G \simeq \mathbf{Z}_p \times \mathbf{Z}_p$ by the Fundamental Theorem of Finite Abelian Groups (**5**). It turns out that (3) is the interesting case.

Suppose that $|G| = 2p$, where $2 < p$ and p is prime. By Cauchy's Theorem there are elements $a, b \in G$ with orders 2 and p respectively (**5**). Let $N = <b>$. Then $|N| = p$; thus $[G : N] = 2$ which means that N is a normal subgroup of G and if H is a subgroup of G such that $N \subseteq H \subseteq G$ then $H = N$ or $H = G$. Now $a \notin N$ since all elements in N, except e have order $p \neq 2$. Therefore N s a proper subgroup of $<a, b>$ which means that $G = <a, b>$ (**5**).

Since N is a normal subgroup of G we have $aba^{-1} = b^\ell$, or equivalently $ab = b^\ell a$, for some $0 \leq \ell < p$. As $aba^{-1} = e$ implies $b = e$ necessarily $1 \leq \ell < p$. Since

$$b = a^2ba^{-2} = a(aba^{-1})a^{-1} = ab^\ell a^{-1} = (aba^{-1})^\ell = (b^\ell)^\ell = b^{\ell^2}$$

it follows that $b^{\ell^2-1} = e$, or $p|(\ell^2 - 1)$, or equivalently $p|(\ell - 1)(\ell + 1)$. Since $0 \leq \ell - 1 < p - 1$ and $2 \leq \ell + 1 \leq p$ it follows that $\ell = 1$ or $\ell = p - 1$ (**5**). In any case there are at most $p{\cdot}2$ expressions of the form that is the expressions of the form $a^{m_1}b^{n_1}\cdots a^{m_r}b^{n_r}$, where $r \geq 1$ and $m_i, n_i \geq 0$; see the solution to Problem 1 (**5**).

Thus G has generators a, b and relations $a^2 = b^4 = e$, and $ab = b^\ell a$ where $\ell = 1$ or $\ell = p-1$. This means there at most two groups of order $2p$ up to isomorphism. We know of two, namely $\mathbf{Z}_{2p}$ and D_p. These are not isomorphic since the first is abelian and the second not. This for (2) there are two possibilities: $G \simeq \mathbf{Z}_{2p}$ and $G \simeq D_p$ (**5**).

4. Page 450, number 24: (**20 points**) First of all we observe that $G = \{\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c, \in \mathbf{Z}_2\}$ is an 8-element subgroup of the group of units (invertible matrices) $\mathrm{M}_3(\mathbf{Z}_2)^\times$ of the ring $\mathrm{M}_3(\mathbf{Z}_2)$. Let $\mathbf{b} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (**5**). The following calculations are omitted: $\mathbf{b}$ has order 4, $\mathbf{a}$ has order 2, $\mathbf{aba} = \mathbf{b}^3$, and $\mathbf{a} \neq \mathbf{b}^2$. Thus $\boxed{\mathbf{b}^4 = \mathbf{a}^2 = e,\ \mathbf{aba} = \mathbf{b}^3}$ (**5**). Since $\mathbf{a} \notin <\mathbf{b}>$ as $\mathbf{b}^2$ is the only element if $<\mathbf{b}>$ of order 2, and $<\mathbf{b}> \subsetneq <\mathbf{a}, \mathbf{b}> \subseteq G$ implies $|<\mathbf{a}, \mathbf{b}>| = 4m$, where $m = 1$ or $m = 2$, then $m \neq 1$ and thus $m = 2$ and consequently G is generated by $\mathbf{a}$ and $\mathbf{b}$ (**5**). Therefore $G \simeq D_4$ (**5**).