# Kaplansky's Ten Hopf Algebra Conjectures

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# 0. Introduction

Hopf algebras were discovered by Heinz Hopf 1941 as structures in algebraic topology **[Hopf]**. Hopf algebras we consider are in the category of vector spaces over a field $\Bbbk$. A few examples: group algebras, enveloping algebras of Lie algebras, and quantum groups. Some quantum groups produce invariants of knots and links.

A general theory of Hopf algebras began in the late 1960s **[Swe]**. In a 1975 publication **[Kap]** Kaplansky listed 10 conjectures on Hopf algebras. These have been the focus of a great deal of research. Some have not been resolved.

We define Hopf algebra, describe examples in detail, and discuss the significance and status of each conjecture. In discussing conjectures more of the nature of Hopf algebras will be revealed.

$\otimes = \otimes_{\Bbbk}$. "f-d" = finite-dimensional. "$n$-d" = $n$-dimensional. *What is a Hopf algebra?*

## 1. A Basic Example and Definition

$G$ is a group, $A = \mathbb{k}[G]$ is the group algebra of $G$ over $\mathbb{k}$. Let $g, h \in G$. The algebra structure:

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|---|---|
| $\mathbb{k}[G] \otimes \mathbb{k}[G] \xrightarrow{m} \mathbb{k}[G]$ | $m(g \otimes h) = gh$ |
| $\mathbb{k} \xrightarrow{\eta} \mathbb{k}[G]$ | $\eta(1_{\mathbb{k}}) = e = 1_{\mathbb{k}[G]}$ |

The <u>co</u>algebra structure:

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|---|---|
| $\mathbb{k}[G] \xrightarrow{\Delta} \mathbb{k}[G] \otimes \mathbb{k}[G]$ | $\Delta(g) = g \otimes g$ |
| $\mathbb{k}[G] \xrightarrow{\epsilon} \mathbb{k}$ | $\epsilon(g) = 1_{\mathbb{k}}$ |

The map which accounts for inverses:

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|---|---|
| $\mathbb{k}[G] \xrightarrow{S} \mathbb{k}[G]$ | $S(g) = g^{-1}$ |

Observe that

$$\Delta(gh) = gh \otimes gh = (g \otimes g)(h \otimes h) = \Delta(g)\Delta(h),$$

$$\epsilon(gh) = 1_{\Bbbk} = 1_{\Bbbk}1_{\Bbbk} = \epsilon(g)\epsilon(h),$$

$$gS(g) = gg^{-1} = e = 1_{\Bbbk}e = \epsilon(g)1_{\Bbbk[G]},$$

and

$$S(g)g = g^{-1}g = e = \epsilon(g)1_{\Bbbk[G]}.$$

Also $\Delta(1_{\Bbbk[G]}) = 1_{\Bbbk[G]} \otimes 1_{\Bbbk[G]}$ and $\epsilon(1_{\Bbbk[G]}) = 1_{\Bbbk}$; thus

$$\boxed{\Delta, \epsilon \text{ are algebra maps}}$$

and $S$ is determined by

$$\boxed{gS(g) = \epsilon(g)1_{\Bbbk[G]} = S(g)g.}$$

We generalize the system $(\Bbbk[G], m, \eta, \Delta, \epsilon, S)$.

**Hopf algebra over** $\mathbb{k}$, a tuple $(A, m, \eta, \Delta, \epsilon, S)$, where $(A, m, \eta)$ is an algebra over $\mathbb{k}$:

| | |
|---|---|
| $A \otimes A \xrightarrow{m} A$ | $m(a \otimes b) = ab$ |
| $\mathbb{k} \xrightarrow{\eta} A$ | $\eta(1_{\mathbb{k}}) = 1_A$ |

$(A, \Delta, \epsilon)$ is a coalgebra over $\mathbb{k}$:

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|---|---|
| $A \xrightarrow{\Delta} A \otimes A$ | $\Delta(a) = a_{(1)} \otimes a_{(2)}$ |
| $A \xrightarrow{\epsilon} \mathbb{k}$ | |

and $\boxed{A \xrightarrow{S} A}$ is an "antipode" where certain axioms are satisfied.

<u>Comments</u>: $\Delta(a) \in A \otimes A$ is usually a **sum** of tensors; thus $\Delta(a) = a_{(1)} \otimes a_{(2)}$ **is a notation**, called the *Heyneman-Sweedler notation*. $\Delta$ is called the *coproduct* and $\epsilon$ the *counit*.

The axioms for a Hopf algebra over $\mathbb{k}$:

$(A, m, \eta)$ is an (associative) algebra over $\Bbbk$

$$\boxed{(ab)c = a(bc), \qquad 1a = a = a1;}$$

$(A, \Delta, \epsilon)$ is a (coassociative) coalgebra over $\Bbbk$

$$\boxed{\begin{array}{l} a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)}, \\ \epsilon(a_{(1)})a_{(2)} = a = a_{(1)}\epsilon(a_{(2)}); \end{array}}$$

$\Delta$ is an algebra map

$$\boxed{\Delta(ab) = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}, \qquad \Delta(1) = 1 \otimes 1;}$$

$\epsilon$ is an algebra map

$$\boxed{\epsilon(ab) = \epsilon(a)\epsilon(b), \qquad \epsilon(1) = 1;}$$ and

$$\boxed{a_{(1)}S(a_{(2)}) = \epsilon(a)1 = S(a_{(1)})a_{(2)}}$$

for all $a, b, c \in A$. From now on $A$ denotes a Hopf algebra over $\Bbbk$.

## 2. Basic Properties and More Definitions

1. A has a unique antipode.

For $g, h \in \mathbb{k}[G]$ observe

$$S(gh) = (gh)^{-1} = h^{-1}g^{-1} = S(h)S(g)$$

and

$$S(e) = e^{-1} = e, \ \ or \ \ S(1_{\mathbb{k}[G]}) = 1_{\mathbb{k}[G]}.$$

Also

$$\epsilon(S(g)) = \epsilon(g^{-1}) = 1_{\mathbb{k}} = \epsilon(g).$$

2. For $a, b \in A$

$$S(ab) = S(b)S(a); \qquad S(1) = 1.$$

Also

$$\Delta(S(a)) = S(a_{(2)}) \otimes S(a_{(1)}), \quad \epsilon(S(a)) = \epsilon(a).$$

Explanation of notation:

$$a_{(2)} \otimes a_{(1)} = \tau(a_{(1)} \otimes a_{(2)}),$$

where $\tau : A \otimes A \longrightarrow A \otimes A$ is given by

$$\tau(a \otimes b) = b \otimes a.$$

3. $a \in A$ *is cocommutative* if $\tau(\Delta(a)) = \Delta(a)$, or $a_{(2)} \otimes a_{(1)} = a_{(1)} \otimes a_{(2)}$. $A$ *is cocommutative* if $\boxed{\tau \circ \Delta = \Delta}$, or if all $a \in A$ are cocommutative.

$\Bbbk[G]$ is cocommutative ($\Delta(g) = g \otimes g$ for $g \in G$).

4. $A$ *is commutative* if $\boxed{m \circ \tau = m}$, or $ba = ab$ for all $a, b \in A$.

$\Bbbk[G]$ is commutative if and only if $G$ is an abelian group.

**Variation on** $ba = ab$; $\qquad ba = qab,\ q \in \Bbbk$.

5. $a \in A$ *is grouplike* if $\Delta(a) = a \otimes a$ and $\epsilon(a) = 1$. G$(A)$ denotes the set of grouplike elements of $A$.

6. G$(A)$ is linearly independent (a coalgebra fact).

G$(\mathbb{k}[G]) = G$.

7. G$(A)$ is a group under multiplication and $S(g) = g^{-1}$ for $g \in$ G$(A)$. Thus if $A$ is f-d then G$(A)$ is a finite group.

8. $(A, m^{op}, \eta, \Delta^{cop}, \epsilon, S)$ is a Hopf algebra over $\mathbb{k}$, where $m^{op} = m \circ \tau$ and $\Delta^{cop} = \tau \circ \Delta$.

9. $(A^*, \Delta^*, \epsilon^*, m^*, \eta^*, S^*)$ is a f-d Hopf algebra over $\mathbb{k}$ if $(A, m, \eta, \Delta, \epsilon, S)$ is.

10. If $A$ is f-d then $S$ is bijective.

11. If $A$ is commutative or cocommutative then $S^2 = \mathrm{I}_A$. In this case $S$ is bijective.

12. If $S$ is bijective then $(A, m^{op}, \eta, \Delta, \epsilon, S^{-1})$, $(A, m, \eta, \Delta^{cop}, \epsilon, S^{-1})$ are Hopf algebras over $\Bbbk$.

13. Let $M, N$ be left $A$-modules (regard $A$ as an algebra). Then $M \otimes N$ is a left $A$-module where $a{\cdot}(m \otimes n) = a_{(1)}{\cdot}m \otimes a_{(2)}{\cdot}n$.

For $g \in \mathrm{G}(A)$ note $g{\cdot}(m \otimes n) = g{\cdot}m \otimes g{\cdot}n$.

14. Let $C$ be a coalgebra over $\Bbbk$. Every f-d subspace of $C$ generates a f-d subcoalgebra.

15. $A$ has simple subcoalgebras, and all are f-d. $\Bbbk 1$, more generally $\Bbbk g$ for $g \in \mathrm{G}(A)$, is a simple subcoalgebra of $A$.

The reader is referred to any basic text on Hopf algebras: **[Swe]**, **[Abe]**, **[Mont]**, **[DNR]**, **[R]**.

## 3. The Enveloping Algebra

Let $L$ be a Lie algebra over $\mathbb{k}$. The enveloping algebra $U(L)$ is a cocommutative Hopf algebra over $\mathbb{k}$ where

$$\boxed{\Delta(\ell) = 1 \otimes \ell + \ell \otimes 1,} \qquad (1)$$

$\epsilon(\ell) = 0$, and $S(\ell) = -\ell$ for $\ell \in L$. An $\ell \in A$ such that (1) holds is *primitive*.

The set of primitives $\mathrm{P}(A)$ of $A$ is a subspace and a Lie algebra under associative bracket.

$A = U(L)$ is *pointed irreducible* meaning $\mathbb{k}1$ is the only simple subcoalgebra of $A$.

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| **Variation on primitive**: $\Delta(\ell) = g \otimes \ell + \ell \otimes h$, where $g, h \in \mathrm{G}(A)$. |

Such an $\ell$ is a *skew primitive element*.

## 4. The 10 Kaplansky conjectures

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| 1. Hopf algebras are free modules over their Hopf subalgebras (under multiplication). |

Let $G$ be a group. The Hopf subalgebras of $\Bbbk[G]$ are the group algebras $\Bbbk[L]$, where $L$ is a subgroup of $G$. True in this case.

When $A$ is f-d the conjecture can be thought of as generalization of **Lagrange's Theorem** for finite groups.

Infinite-dimensional counterexamples: Takeuchi in 1972 **[Tak]**, Oberst and Schneider in 1974 **[ObSch]**, the speaker 1980 **[Rad]**, Schneider in 1981. **[Sch1]**.

Proved when $A$ is f-d by Nichols and Zoeller **[NicZel1]** in 1989. Implications when $A$ is f-d:

If $B$ is a Hopf subalgebra of $A$ then Dim($B$) divides Dim($A$).

$|\mathrm{G}(A)| = \mathrm{Dim}(\Bbbk[\mathrm{G}(A)])$ divides Dim($A$).

Suppose $B$ is a Hopf subalgebra of $A$. Then $A$ semisimple implies $B$ is also.

Nichols and Zoeller used representation theory of algebras to establish relative Hopf modules are free when $A$ is f-d.

A relative Hopf module is a structure $(M, \mu, \rho)$, where

$$\mu : B \otimes M \longrightarrow M$$

is a left $B$-module and

$$\rho : M \longrightarrow A \otimes M$$

is a left $A$-comodule, which have a certain compatibility. With $(b \otimes b')(a \otimes m) = ba \otimes b'{\cdot}m$,

$$\boxed{\rho(b{\cdot}m) = \Delta(b)\rho(m)}.$$

When $B = A$ then $(M, \mu, \rho)$ is a Hopf module. Such are always free with $A$-basis any linear basis of

$$M^{co\ inv} = \{m \in M \mid \rho(m) = 1 \otimes m\}.$$

Freeness of Hopf modules is elementary in that no representation theory needed.

Implications of freeness of Hopf modules:

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| If $A$ contains a non-zero f-d left ideal then $A$ is f-d. |

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| If $A$ is semisimple then $A$ is f-d. |

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| If $A$ is f-d then $A$ contains a 1-d ideal. |

A definition for our next conjecture.

**Definition**: A coalgebra is admissible if there is an algebra structure which makes it a Hopf algebra.

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| 2. A coalgebra is admissible iff every f-d subspace lies in a f-d admissible subcoalgebra. |

Assume $\Bbbk$ has characteristic 0. If $0 \neq \ell \in \mathrm{P}(A)$ the Hopf subalgebra generated by $\ell$ is $\Bbbk[\ell]$, the free $\Bbbk$-algebra on $\Bbbk\ell$.

$\Bbbk[x]$ is a Hopf algebra where $x$ is primitive. Any Hopf subalgebra contains $x$ or is $\Bbbk 1$. Thus:

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| The Hopf subalgebras of $\Bbbk[x]$ are $\Bbbk 1$ and $\Bbbk[x]$. |

Hence the conjecture is false. This example is due to Larson. Note $\Bbbk[x] = U(\Bbbk x)$.

Generally the Hopf subalgebras of $U(L)$ are $U(M)$, where $M$ is a Lie subalgebra of $L$, when the characteristic of $\Bbbk$ is zero.

3. A Hopf algebra in characteristic 0 has no non-zero nilpotent elements.

Let $G$ be a finite group, $\Lambda = \sum_{g \in G} g \in \mathbb{k}[G]$. Then $g\Lambda = \Lambda$, or $g\Lambda = \epsilon(g)\Lambda$, for all $g \in G$. Thus

$$a\Lambda = \epsilon(a)\Lambda \tag{2}$$

for all $a \in \mathbb{k}[G]$. Since

$$\epsilon(\Lambda) = \sum_{g \in G} \epsilon(g) = |G| 1_{\mathbb{k}} :$$

**Maschke's Theorem** Let $G$ be a finite group. Then $\mathbb{k}[G]$ is semisimple if and only if $\epsilon(\Lambda) \neq 0$.

Let $A$ be f-d. Then $A$ contains a non-zero $\Lambda$ which satisfies (2) for all $a \in A$. Further

$A$ is semisimple if and only if $\epsilon(\Lambda) \neq 0$.

If $A$ is not semisimple $\Lambda^2 = \epsilon(\Lambda)\Lambda = 0\Lambda = 0$; thus $\Lambda$ is a non-zero nilpotent element.

There are f-d Hopf algebras over $\mathbf{C}$ which are not semisimple. Thus the conjecture is false.

**Example 1**: $A$ as an algebra is generated by $a, x$ subject to the relations

$$a^2 = 1, \quad x^2 = 0, \quad xa = -ax$$

and as a coalgebra its structure is given by

$$\Delta(a) = a \otimes a, \quad \Delta(x) = x \otimes a + 1 \otimes x$$

and

$$\epsilon(a) = 1 \quad \epsilon(x) = 0.$$

$$S(a) = a, \text{ and } S(x) = ax.$$

Note

$$S^2(x) = S(ax) = S(x)S(a) = (ax)a = -x;$$

thus $S^2 \neq \mathrm{I}_A$. $\mathrm{Dim}(A) = 4$. We can take

$$\Lambda = (1 + a)x.$$

$A$ is not semisimple since

$$\epsilon(\Lambda) = \epsilon(1 + a)\epsilon(x) = \epsilon(1 + a)0 = 0.$$

4. An element $x$ in a Hopf algebra is in its center if $a_{(1)}xS(a_{(2)}) = \epsilon(a)x$ for all $a \in A$.

$x \in \Bbbk[G]$ is central if and only if

$$gxg^{-1} = x \ \ (= \epsilon(g)x)$$

for all $g \in G$. The formula

$$gxg^{-1} = \epsilon(g)x$$

translates to

$$a{\cdot}x := a_{(1)}xS(a_{(2)}) = \epsilon(a)x$$

for all $a, x \in A$. The conjecture rewritten:

$x \in A$ is central if and only if $a{\cdot}x = \epsilon(a)x$

for all $a \in A$. If $x$ is central then

$$a{\cdot}x = a_{(1)}S(a_{(2)})x = a_{(1)}\epsilon(a_{(2)})1x = \epsilon(a)x.$$

Observe $a{\cdot}x$ defines a left $A$-module action on itself.

For $g, x \in \Bbbk[G]$ the formula

$$gx = (gxg^{-1})g = gxS(g)g$$

translates to

$$\begin{aligned} ax &= a_{(1)}xS(a_{(2)(1)})a_{(2)(2)} \\ &= a_{(1)(1)}xS(a_{(1)(2)})a_{(2)} \\ &= (a_{(1)}\cdot x)a_{(2)}; \end{aligned}$$

thus the commutation relation

$$\boxed{ax = (a_{(1)}\cdot x)a_{(2)}}$$

holds for all $a, x \in A$.

If $a\cdot x = \epsilon(a)x$ for all $a \in A$ then

$$ax = \epsilon(a_{(1)})xa_{(2)} = x\epsilon(a_{(1)})a_{(2)} = xa$$

for all $a \in A$ and therefore $x$ is central.

The conjecture is true.

From now on suppose $A$ is f-d.

5. If $A$ or $A^*$ is semisimple then $S^2 = \mathrm{I}_A$.

Established by Larson and the speaker [**RL1**, **RL2**] when $\Bbbk$ has characteristic 0 in 1988.

Over any field

$$\mathrm{Tr}(S^2) = \mathrm{Dim}(A)\mathrm{Tr}(S^2|Ax), \qquad (3)$$

where $x \in A$ is defined by $p(x) = \mathrm{Tr}(\ell(p))$ for all $p \in A^*$ and $\ell(p)(q) = pq$ for all $q \in A^*$.

First assume $A$ <u>and</u> $A^*$ are semisimple. Then $S^4 = \mathrm{I}_A$. Now assume the characteristic of $\Bbbk$ is 0. Then $S^2$ is diagonalizable and eigenvalues are $\pm 1$. But $-1$ is not an eigenvalue of $S^2$ by (3). Thus $S^2 = \mathrm{I}$. Now $A$ or $A^*$ semisimple implies $A$ and $A^*$ are in characteristic 0.

The characteristic $p > 0$ case is still open.

Etingof and Gelaki showed in 1998 if $A$ <u>and</u> $A^*$ are semisimple then $S^2 = \mathrm{I}_A$ by “lifting” to the characteristic 0 case [**EtGel1**].

6. The size of matrices occurring in any matrix constituent of $A$ divides $\mathrm{Dim}(A)$.

The conjecture is interpreted:

If $A$ is a f-d Hopf algebra over an algebraically closed field $\Bbbk$ and $M$ is a simple $A$-module then $\mathrm{Dim}(M)|\mathrm{Dim}(A)$.

True when $A = \Bbbk[G]$ is semisimple, false for group algebras in general.

*Assume $A$ is semisimple* and that $M$ is a simple $A$-module. Then $\mathrm{Dim}(M)|\mathrm{Dim}(A)$ when:

(1) $\mathrm{Dim}(M) = 2$, due to Nichols and Zoeller [**NicZel2**] in 1996;

(2) $A = D(B)$, where $B$ is a f-d semisimple, due to Etingof and Gelaki [**EtGel2**] in 1998;

(3) $A$ is cosemisimple and quasitriagular, due to Etingof and Gelaki [**EtGel2**] 1998;

(4) Dim($A$) is a prime power, this is due to Montgomery and Witherspoon [**MW**] in 1998.

Regarding (2), [**EtGel2**] builds on important work of Zhu [**Zhu1**]. He obtains partial results for $D(B)$.

Regarding (3), every f-d quasitriangular Hopf algebra $A$ is a quotient of $D(A)$. Therefore (2) implies (3).

Recent progress on conjecture six is through classification of semisimple Hopf algebras. We single out Natale's 2007 paper [**Nat**].

There are many contributors to classifiaction. See the 2008 survey by Masuoka [**Mas**].

7. If $A$ and $A^*$ are both semisimple then the characteristic of $\Bbbk$ does not divide $\mathrm{Dim}(A)$.

Settled in 1988. Let $\Lambda \in A$ satisfy (2), that is

$$a\Lambda = \epsilon(a)\Lambda$$

for all $a \in A$ and $\lambda \in A^*$ satisfy

$$\lambda p = p(1)\lambda$$

for all $p \in A^*$. We may assume $\lambda(\Lambda) = 1$. Then

$$\mathrm{Tr}(S^2) = \epsilon(\Lambda)\lambda(1). \tag{4}$$

Both $A$ and $A^*$ are semisimple if and only if $\mathrm{Tr}(S^2) \neq 0$.

Now we use (4) and (3), which is

$$\mathrm{Tr}(S^2) = \mathrm{Dim}(A)\mathrm{Tr}(S^2|Ax),$$

to prove the conjecture. Equation (4) is is found in the 1988 paper [**LR1**] by Larson and the speaker.

When $\Bbbk = \mathbf{Z}_p$ note $\Bbbk[\mathbf{Z}_p]^*$ is semisimple, $\Bbbk[\mathbf{Z}_p]$ is not, and $\mathrm{Dim}(\Bbbk[\mathbf{Z}_p]) = p$.

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| 8. If Dim($A$) is prime then $A$ is commutative and cocommutative. |

We may assume that $\Bbbk$ is algebraically closed. Suppose Dim($A$) is prime. Then $S^4 = \mathrm{I}_A$.

Assume $\Bbbk$ has characteristic 0. Tr($S^2$) $\neq$ 0. Thus $A$ (and also $A^*$) is semisimple. We use Zhu's class equation in the 1994 paper [**Zhu2**]. $\mathrm{C}_\Bbbk(A) \subseteq A^*$ is the span of the characters of $A$. It is a semisimple algebra. For semisimple $A$:

**Theorem 1** *Let* $\{e_1, \ldots, e_n\}$ *be a complete set of orthogonal idempotents for* $\mathrm{C}_\Bbbk(A)$*. Then*

$$\mathrm{Dim}(A) = \sum_{i=1}^{n} \mathrm{Dim}(e_i A^*),$$

$\mathrm{Dim}(e_i A^*) | \mathrm{Dim}(A)$ *for all* $i$*, and* $\mathrm{Dim}(e_i A^*) = 1$ *for some* $i$*.*

There is a bijective correspondence between G($A$)∩Z($A$) and the set of 1-dimensional $e_i A^*$'s. Thus $p =$ Dim($A$) prime implies $A = \Bbbk[\mathbf{Z}_p]$.

Zhu established the conjecture in this case in 1993 in [**Zhu1**]. The class equation is based on work of Kac [**Kac**].

The characteristic $p > 0$ case has not been completely decided.

Etingof and Gelaki showed $A$ is commutative and cocommutative when both $A$ and $A^*$ are semisimple by "lifting" to the characteristic zero case in 1998 [**EtGel2**].

For the two remaining conjectures suppose the characteristic of $\Bbbk$ does not divide $\mathrm{Dim}(A)$.

9. The radicals of $A$ and $A^*$ have the same dimension.

We reformulate the conjecture. $A_0$ is the sum of the simple subcoalgebras of $A$.

$$\text{Rad}(A^*) = A_0^{\perp} := \{p \in A^* \mid p(A_0) = (0)\}.$$

Reformulation: $\boxed{\text{Dim}(A_0) = \text{Dim}({A^*}_0)}$

**Example 2**: Consider $A$ of Example 1. $A_0 = \Bbbk 1 \oplus \Bbbk a$ and thus $\text{Dim}(A_0) = 2$. The Drinfel'd double, a very special quantum group,

$$D(A) = A^{*\,cop} \otimes A = A \otimes A$$

as a coalgebra. Thus $\text{Dim}(D(A)_0) = 4$ since

$$D(A)_0 = (A \otimes A)_0 = A_0 \otimes A_0.$$

As $D(A)$ has a 2-d simple module, $D(A)^*$ has a 4-d simple subcoalgebra $C$. Therefore the conjecture is false since ${D(A)^*}_0 \supseteq \Bbbk 1 \oplus C$ and hence $\text{Dim}(D(A)_0) = 4 < \text{Dim}({D(A)^*}_0)$.

Counterexamples were given by Schneider in 1995 [**Sch2**] and Sommerhäuser in 1998 [**Som**].

10. For a given dimension there are only finitely many different isomorphism types of Hopf algebras.

Established for semisimple <u>co</u>semisimple Hopf algebras by Ştefan in 1997 [**Stef**].

Generally false, shown by Andruskiewitsch and Schneider in 2000 [**AS**], by Beattie, Dăscălescu, and Grünenfelder in 1999 [**BDG**], by Gelaki in 1998 [**Gel**], and by Müller in 2000 [**Mue**]. These were independent efforts.

We describe the family of $H_q(a)$'s, where $a \in \mathbb{k}$, of [**BDG**]. $\mathrm{Dim}(H_q(a)) = p^4$, where $p$ is an odd prime. Assume $q \in \mathbb{k}$ is a primitive $p^{th}$ root of unity and $\mathbb{k}$ is infinite.

**Example 3**: $H_q(a)$ is generated as a $\Bbbk$-algebra by $c, x, y$ subject to the relations

$$c^{n^2} = 1, \quad x^n = c^n - 1, \quad y^n = c^n - 1,$$

$$xc = q^{-1}cx, \quad yc = qcy, \quad yx = qxy + a(c^2 - 1).$$

As a $\Bbbk$-coalgebra $H_q(a)$ is determined by

$$\Delta(c) = c \otimes c, \quad \Delta(z) = c \otimes z + z \otimes 1,$$

$$\epsilon(c) = 1, \quad \epsilon(z) = 0,$$

where $z = x, y$.

$H_q(b) \simeq H_q(a)$ if and only if $b = ua$ for some $p^{th}$ root of unity $u \in \Bbbk$.

**Remark**: As an algebra $H_q(a)$ is a quotient of iterated Ore extensions.

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A fuller account of the conjectures is found in Sommerhäuser's excellent 2000 expository paper [**Som**]. Appended is an expanded list of references.

# References

[**Abe**] Abe, Eiichi. Hopf algebras. Cambridge Tracts in Mathematics, **74**. Cambridge University Press, Cambridge-New York, 1980. xii+284 pp.

[**AS**] Andruskiewitsch, N. and Schneider, H.-J. Lifting of Nichols algebras of type $A_2$ and pointed Hopf algebras of order $p^4$. Hopf algebras and quantum groups (Brussels, 1998), in *Lecture Notes in Pure and Appl. Math.* **209**, Dekker, New York, 2000. pp. 1-14.

[**BDG**] Beattie, M., Dăscălescu, S. and Grünenfelder, L. Constructing pointed Hopf algebras by Ore extensions, *J. Algebra* **225**, 2000. pp. 743-770.

[**CR 2006**] Curtis, C. W. and Reiner, I. *Representation theory of finite groups and associative algebras*. (Reprint of the 1962 original.) AMS Chelsea Publishing, Providence, RI, 2006. xiv+689.

[**DNR**] Dascalescu, Sorin; Nastasescu, Constantin; Raianu, Serban. Hopf algebras. An introduction. Monographs and Textbooks in Pure and Applied Mathematics, **235**. Marcel Dekker, Inc., New York, 2001. x+401 pp.

[**Drinfel'd 1989**] Almost cocommutative Hopf algebras, (Russian) *Algebra i Analiz* **1** (1989), pp. 30–46, translation in *Leningrad Math. J.* **1**, 1990. pp. 321-342.

[**Drinfel'd 1986**] Drinfel'd, V. G. Quantum groups, in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2* (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987. pp. 798-820.

[**EtGel1**] Etingof, Pavel; Gelaki, Shlomo. On finite-dimensional semisimple and cosemisimple Hopf algebras in positive characteristic. *Internat. Math. Res. Notices* 1998. pp. 851–864.

[**EtGel2**] Etingof, Pavel; Gelaki, Shlomo. Some properties of finite-dimensional semisimple Hopf algebras. *Math. Res. Lett.* **5**, 1998. pp. 191-197.

[**Gel**] Gelaki, S. Pointed Hopf algebras and Kaplansky's 10th conjecture, *J. Algebra* **209**, 1998. pp. 635-657.

[**Haz 2008**] M. Hazewinkel. Niceness Theorems (2008) arXiv:0810.5691

[**Humph1972**] Humphreys, J. E. *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics **9**. Springer-Verlag, New York-Berlin, 1972. xii+169 pp.

[**Hey-Swe 1969**] Heyneman, Robert G.; Sweedler, Moss E. Affine Hopf algebras. II. *J. Algebra* **13**, 1969. pp. 192–241.

[**Hey-Swe 1970**] Heyneman, Robert G.; Sweedler, Moss E. Affine Hopf algebras. II. *J. Algebra* **16**, 1970. pp. 271–297.

[**Hopf**] Hopf, Heinz. Über die Topologie der Gruppen- Mannigfaltigkeiten und ihre Verallgemeinerungen. *Ann. of Math.* **42**, (1941). pp. 22–52.

[**Kac**] Kac, G. I. Certain arithmetic properties of ring groups, *Functional Anal. Appl.* **6**, 1972. pp. 158-160.

[**Kap**] Kaplansky, Irving. Bialgebras. Lecture Notes in Mathematics. Department of Mathematics, University of Chicago, Chicago, Ill., 1975. iv+57 pp.

[**Kassel-Rosso-Tur 1997**] Kassel, Christian; Rosso, Marc; Turaev, Vladimir. Quantum groups and knot invariants. Panoramas et Synthses Panoramas and Syntheses **5**. Socit Mathmatique de France, Paris, 1997. vi+115 pp.

[**LR1**] Larson, Richard G.; Radford, David E. Finite-dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple. *J. Algebra* **117**, 1988. pp. 267–289.

[**LR2**] Larson, Richard G.; Radford, David E. Semisimple cosemisimple Hopf algebras. *Amer. J. Math.* **110** 1988. pp. 187–195.

[**LarSwe 1969**] Larson, R. G. and Sweedler, M. E. An associative orthogonal bilinear form for Hopf algebras, *Amer. J. Math.* **91**, 1969. pp. 75–94.

[**Mas**] Masuoka, A. Classification of semisimple Hopf algebras, in *Handbook of algebra* **5**, Elsevier/North-Holland, Amsterdam, 2008. pp. 429455.

[**Mont**] Montgomery, Susan. Hopf algebras and their actions on rings. CBMS Regional Conference Series in Mathematics, 82. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993. xiv+238 pp.

[**MW**] Montgomery, S. and Witherspoon, S. J. Irreducible representations of crossed products, *J. Pure Appl. Algebra* **129**, 1998. pp. 315-326.

[**Mue**] Müller, E. Finite subgroups of the quantum general linear group, *Proc. London Math. Soc. (3)* **81**, 2000. pp. 190-210.

[**Nat**] Natale, Sonia. *Semisolvability of semisimple Hopf algebras of low dimension*, Mem. Amer. Math. Soc. 186, **874**, 2007. viii+123 pp.

[**NicZel1**] Nichols, Warren D.; Zoeller, M. Bettina. A Hopf algebra freeness theorem. *Amer. J. Math.* **111**, 1989. pp. 381–385.

[**NicZel2**] Nichols, Warren D.; Richmond, M. Bettina. The Grothendieck group of a Hopf algebra. *J. Pure Appl. Algebra* **106** 1996. pp. 297-306.

[**ObSch**] Oberst, U. and Schneider, H.-J. Untergruppen formeller Gruppen von endlichem Index (German), *J. Algebra* **31**, 1974. pp. 10–44.

[**Rad 1993**] Radford, D. E. Minimal quasitriangular Hopf algebras, *J. Algebra* **157**, 1993. pp. 285-315.

[**R**] Radford, D. E. Hopf Algebras. World Scientific Press, 2012 xxii + 559 pp.

[**Rad**] Radford, D. E. On an analog of Lagrange's theorem for commutative Hopf algebras, *Proc. Amer. Math. Soc.* **79**, 1980. pp. 164–166.

[**Sch1**] Schneider, H.J. Zerlegbare Untergruppen affiner Gruppen, (German) *Math. Ann.* **255**, 1981. pp. 139-158.

[**Sch2**] Schneider, Hans-Jürgen. Lectures on Hopf algebras. Notes by Sonia Natale. *Trabajos de Matemtica [Mathematical Works]* **31/95**. Universidad Nacional de Crdoba, Facultad de Matemtica, Astronoma y Fsica, Crdoba, 1995. 58 pp.

[**Som**] Sommerhäuser, Yorck. On Kaplansky's conjectures. *Interactions between ring theory and representations of algebras (Murcia)*, 393412, Lecture Notes in Pure and Appl. Math., **210**, Dekker, New York, 2000.

[**Stef**] Ştefan, D̦. The set of types of $n$-dimensional semisimple and cosemisimple Hopf algebras is finite, *J. Algebra* **193**, 1997. pp. 571-580.

[**Swe 1969**] Sweedler, Moss Eisenberg. Integrals for Hopf algebras. *Ann. of Math.* **89**, 1969. pp. 323–335.

[**Swe**] Sweedler, Moss E. Hopf algebras. Mathematics Lecture Note Series W. A. Benjamin, Inc., New York 1969 vii+336 pp.

[**Taft 1971**] Taft, Earl J. The order of the antipode of finite-dimensional Hopf algebra. *Proc. Nat. Acad. Sci. U.S.A.* **68** 1971. 2631–2633.

[**Tak**] Takeuchi, M. Relative Hopf modules—equivalences and freeness criteria, *J. Algebra* **60**, 1979. pp. 452–471.

[**Zhu1**] Zhu, Y. Quantum double construction of quasitriangular Hopf algebras and Kaplansky's conjecture, preprint 1993.

[**Zhu2**] Zhu, Y. Hopf algebras of prime dimension, *Internat. Math. Res. Notices*, 1994. pp. 53-59.