The Mathematics for Essay 2

You may assume basic facts about geometric sums and series for Essay 2. Let \( r \) be any real number and let \( n \) be a non-negative integer. The sum

\[
1 + r + r^2 + \cdots + r^n
\]

is a geometric sum and the infinite series

\[
1 + r + r^2 + \cdots + r^n + \cdots
\]

is a geometric series.

Suppose further that \( r \neq 1 \). Then the geometric sum (1) can be computed by the formula

\[
1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.
\]

This fact, which you may assume, is easily proved proved by mathematical induction.

Now suppose that \( |r| < 1 \). Then \( \lim_{n \to \infty} r^n = 0 \) which means the geometric series (2) converges to \( \frac{1}{1 - r} \) by the preceding equation. We write

\[
1 + r + r^2 + \cdots + r^n + \cdots = \frac{1}{1 - r}
\]

to indicate that the series converges and to designate the limit of the sequence of partial sums.

Your essay will involve the geometric series

\[
1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots.
\]

Since \( |\frac{1}{2}| < 1 \), it follows by (3) that (4) converges and \( 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots = 2 \). The DeLuxe blocks are cubes with side lengths 1, \( \frac{1}{2} \), \( \frac{1}{3} \), \( \frac{1}{5} \), \( \frac{1}{9} \), \( \frac{1}{16} \), \( \cdots \) Your essay involves analyzing the sum of their side lengths

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{16} + \cdots.
\]

The preceding series is called the harmonic series. Think of the terms of the geometric series (4) as markers for grouping terms of the harmonic series as follows:

\[
1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + (\frac{1}{9} + \cdots + \frac{1}{16}) + \cdots.
\]
We will find an overestimate and an underestimate for the sum of the terms in each of the parenthesized groups. You will see a pattern emerging in our calculations:

\[
1 = \frac{1}{2} + \frac{1}{2} > \frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2},
\]

\[
1 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} > \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2},
\]

\[
1 = \frac{1}{8} + \cdots + \frac{1}{8} = 8\left(\frac{1}{8}\right) > \frac{1}{9} + \cdots + \frac{1}{16} > \frac{1}{16} + \cdots + \frac{1}{16} = 8\left(\frac{1}{16}\right) = \frac{1}{2}.
\]

Using (5) and our underestimates, we see that

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{16} + \cdots
\]

\[
= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots
\]

\[
> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots.
\]

Thus the partial sums of the harmonic series grow without bound which is expressed by

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty.
\]

Below is a formal proof of the the fact that the sums of terms in parenthesized groupings lie between \(\frac{1}{2}\) and 1. You should not include the proof in your essay; the mathematics of your essay is to be treated informally. Observe that the terms of a parenthesized group are given by \(\frac{1}{2n+1}, \ldots, \frac{1}{2n+1}\) for some \(n \geq 1\).

**Lemma 1** Let \(n\) be a positive integer. Then \(\frac{1}{2} < \frac{1}{2n+1} + \cdots + \frac{1}{2n+1} < 1\).  

**Proof:** Since \(2^{n+1} = 2^n + 2^n\) the sum in the statement of the lemma has \(2^n\) terms. Each term has the form \(\frac{1}{2n+\ell}\) for some \(1 \leq \ell \leq 2^n\) and thus satisfies

\[
\frac{1}{2n+1} \leq \frac{1}{2n+\ell} \leq \frac{1}{2^n}.
\]

At least one of the terms is larger than \(\frac{1}{2}\), and one at least one is smaller than 1. Therefore

\[
\frac{1}{2} = 2^n\left(\frac{1}{2n+1}\right) < \frac{1}{2n+1} + \cdots + \frac{1}{2n+1} < 2^n\left(\frac{1}{2n}\right) = 1.
\]

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