Hopf Algebras, an Overview

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0. Introduction

Hopf algebras are named after Heinz Hopf who discovered them the last century in the context of algebraic topology [Hopf 1941]. They arise in many areas of mathematics. Group algebras, enveloping algebras of Lie algebras, and quantum groups are examples of Hopf algebras. Certain Hopf algebras give rise to invariants of knots and links. We will describe Hopf algebras, discuss basic examples and fundamental results, and trace the development of the theory.

The Hopf algebras we discuss here are technically different from those coming from algebraic topology. See the very interesting discussion in [Haz 2008].

Objects are vector spaces over a field $\mathbb{k}$ and maps are $\mathbb{k}$-linear. $\otimes = \otimes_{\mathbb{k}}$. “f-d” = finite-dimensional.
1. A Basic Example and Definitions

$G$ is a group and $A = \mathbb{k}G$ is the group algebra of $G$ over $\mathbb{k}$. Let $g, h \in G$. The algebra structure:

\[
\begin{align*}
\mathbb{k}G \otimes \mathbb{k}G & \xrightarrow{m} \mathbb{k}G \\
\mathbb{k} & \xrightarrow{\eta} \mathbb{k}G
\end{align*}
\]

$m(g \otimes h) = gh \\
\eta(1_\mathbb{k}) = e = 1_{\mathbb{k}G}$

The coalgebra structure:

\[
\begin{align*}
\mathbb{k}G & \xrightarrow{\Delta} \mathbb{k}G \otimes \mathbb{k}G \\
\mathbb{k}G & \xrightarrow{\epsilon} \mathbb{k}
\end{align*}
\]

$\Delta(g) = g \otimes g \\
\epsilon(g) = 1_\mathbb{k}$

The map which accounts for inverses:

\[
\begin{align*}
\mathbb{k}G & \xrightarrow{S} \mathbb{k}G \\
S(g) = g^{-1}
\end{align*}
\]
Observe that
\[ \Delta(gh) = gh \otimes gh = (g \otimes g)(h \otimes h) = \Delta(g)\Delta(h), \]
\[ \epsilon(gh) = 1_k^1 = 1_k 1_k = \epsilon(g)\epsilon(h), \]
\[ S(gh) = (gh)^{-1} = h^{-1}g^{-1} = S(h)S(g), \]
\[ gS(g) = gg^{-1} = 1_{kG} = 1_k 1_{kG} = \epsilon(g)1_{kG}, \]
and
\[ S(g)g = g^{-1}g = 1_{kG} = \epsilon(g)1_{kG}. \]

In particular, \( \Delta, \epsilon \) are algebra maps and \( S \) is determined by

\[ gS(g) = \epsilon(g)1_{kG} = S(g)g. \]

We generalize the system \((kG, m, \eta, \Delta, \epsilon, S)\).
A **Hopf algebra over** $k$ is a tuple $(A, m, η, ∆, ϵ, S)$, where $(A, m, η)$ is an algebra over $k$:

$A \otimes A \xrightarrow{m} A \quad m(a \otimes b) = ab$

$k \xrightarrow{η} A \quad η(1_k) = 1_A$

$(A, ∆, ϵ)$ is a **coalgebra over** $k$:

$A \xrightarrow{∆} A \otimes A \quad ∆(a) = a_{(1)} \otimes a_{(2)}$

$A \xrightarrow{ϵ} k$

and $A \xrightarrow{S} A$ is an "antipode" where certain axioms are satisfied.

**Comments**: $∆(a) \in A \otimes A$ is usually a *sum* of tensors; thus $∆(a) = a_{(1)} \otimes a_{(2)}$ **is a notation**, called the Heyneman-Sweedler notation. $∆$ is called the **coproduct** and $ϵ$ the **counit**.

The axioms for a Hopf algebra:
\[(A, m, \eta)\] is an (associative) algebra:

\[(ab)c = a(bc),\quad 1a = a = a1\]

\[(A, \Delta, \epsilon)\] is a (coassociative) coalgebra:

\[a(1)(1) \otimes a(1)(2) \otimes a(2) = a(1) \otimes a(2)(1) \otimes a(2)(2),\]
\[\epsilon(a(1))a(2) = a = a(1)\epsilon(a(2))\]

\[\Delta\] is an algebra map:

\[\Delta(ab) = a(1)b(1) \otimes a(2)b(2),\quad \Delta(1) = 1 \otimes 1,\]

\[\epsilon\] is an algebra map:

\[\epsilon(ab) = \epsilon(a)\epsilon(b),\quad \epsilon(1) = 1,\]

and

\[a(1)S(a(2)) = \epsilon(a)1 = S(a(1))a(2)\]

for all \(a, b \in A\). From now on \(A\) denotes a Hopf algebra over \(k\).
2. Basic Properties and More Definitions

1. A has a unique antipode and

\[
S(ab) = S(b)S(a), \quad S(1) = 1. \quad \text{Also}
\]

\[
\Delta(S(a)) = S(a_2) \otimes S(a_1), \quad \epsilon(S(a)) = \epsilon(a).
\]

2. \(a \in A\) is cocommutative if \(a_1 \otimes a_2 = a_2 \otimes a_1\); \(A\) is cocommutative if all \(a \in A\) are. \(kG\) is cocommutative. \(A\) is commutative if \(ab = ba\) for all \(a, b \in A\). \(kG\) is commutative iff \(G\) is.

3. \(a \in A\) is grouplike if \(\Delta(a) = a \otimes a\) and \(\epsilon(a) = 1\). \(1 \in A\) is grouplike. The set \(G(A)\) of grouplike elements of \(A\) is linearly independent (coalgebra fact). \(G(kG) = G\).

4. \(G(A)\) is a group under multiplication and \(S(g) = g^{-1}\) for \(g \in G(A)\). Thus if \(A\) is f-d then \(G(A)\) is a finite group.
5. Let $M, N$ be left $A$-modules (regard $A$ as an algebra). Then $M \otimes N$ is a left $A$-module where

$$a \cdot (m \otimes n) = a(1) \cdot m \otimes a(2) \cdot n.$$ 

For $g \in G \subseteq \mathbb{k}G$ note $g \cdot (m \otimes n) = g \cdot m \otimes g \cdot n$.

6. If $S$ is bijective then $(A, m^{op}, \eta, \Delta, \epsilon, S^{-1})$, $(A, m, \eta, \Delta^{cop}, \epsilon, S^{-1})$ are Hopf algebras, where $m^{op}(a \otimes b) = ba$ and $\Delta^{cop}(a) = a(2) \otimes a(1)$. $(A, m^{op}, \eta, \Delta^{cop}, \epsilon, S)$ is a Hopf algebra.

7. If $A$ is f-d then $S$ is bijective.

8. If $(A, m, \eta, \Delta, \epsilon, S)$ is f-d $(A^*, \Delta^*, \epsilon^*, m^*, \eta^*, S^*)$ is a f-d Hopf algebra over $\mathbb{k}$.

9. Let $C$ be a (coassociative) coalgebra. Every f-d subspace of $C$ generates a f-d subcoalgebra. Thus $A$ has simple subcoalgebras, and all are f-d. $\mathbb{k}1$, more generally $\mathbb{k}g$ for $g \in G(A)$, is a simple subcoalgebra of $A$.

The reader is referred to any basic text on Hopf algebras: [Swe 1967], [Abe 1980], [Mont 1993], [D–N–Rai 2001].
3. The Enveloping Algebra

$L$ is a Lie algebra over $\mathbb{k}$. The enveloping algebra $U(L)$ is a cocommutative Hopf algebra over $\mathbb{k}$ where

$$\Delta(\ell) = 1 \otimes \ell + \ell \otimes 1, \quad (1)$$

$\epsilon(\ell) = 0$, and $S(\ell) = -\ell$ for $\ell \in L$. An $\ell \in A$ such that (1) holds is primitive. The set of primitives $P(A)$ of $A$ is a subspace and a a Lie algebra under associative bracket. $A = U(L)$ is pointed irreducible meaning $\mathbb{k}1$ is the only simple subcoalgebra of $A$.

Assume $\mathbb{k}$ has characteristic 0. Then $P(U(L))) = L$ and the Milnor-Moore Theorem [Mil–Mo 1965] characterizes the finitely generated graded pointed irreducible Hopf algebras over $\mathbb{k}$ as the enveloping algebras of f-d Lie algebras over $\mathbb{k}$. See [Haz 2008] also.
4. Affine Algebraic Groups

Coalgebras $C$ over $\mathbb{k}$ provide many examples of algebras. The linear dual $C^* = \text{Hom}_k(C, \mathbb{k})$ is an algebra over $\mathbb{k}$ with convolution product:

\[
1_{C^*} = \epsilon \quad \text{and} \quad ab(c) = a(c(1))b(c(2))
\]

for all $a, b \in C^*$ and $c \in C$.

**Example 1** $C$ has basis $c_0, c_1, c_2, \ldots$ and

\[
\Delta(c_n) = \sum_{\ell=0}^{n} c_{n-\ell} \otimes c_\ell, \quad \epsilon(c_n) = \delta_{n,0}.
\]

As $(ab)(c_n) = \sum_{\ell=0}^{n} a(c_{n-\ell})b(c_\ell)$,

\[
C^* \simeq \mathbb{k}[[x]], \quad a \mapsto \sum_{n=0}^{\infty} a(c_n)x^n.
\]

Example 1 suggests coalgebra connections with combinatorics [Rom – Rota 1978], [Rota 1978].
Example 2 Let $n \geq 1$ and $C(n, \mathbb{k})$ have basis $x_{i,j}$, $1 \leq i, j \leq n$ and

$$\Delta(x_{i,j}) = \sum_{\ell=1}^{n} x_{i,\ell} \otimes x_{\ell,j}, \quad \epsilon(x_{i,j}) = \delta_{i,j}.$$  

As $(ab)(x_{i,j}) = \sum_{\ell=1}^{n} a(x_{i,\ell})b(x_{\ell,j}),$

$C^* \simeq M(n, \mathbb{k}), \quad a \mapsto (a_{i,j}), \quad \text{where } a_{i,j} = a(x_{i,j}).$

We now continue. $B = S(C(n, \mathbb{k}))$ is the free commutative $\mathbb{k}$-algebra on $C(n, \mathbb{k})$. Now let $\Delta : B \rightarrow B \otimes B$ and $\epsilon : B \rightarrow \mathbb{k}$ be the algebra maps determined on $x_{i,j}$ as in Example 2. $Alg_{\mathbb{k}}(B, \mathbb{k})$ is closed under the convolution product, contains $\epsilon$, and

$$Alg_{\mathbb{k}}(B, \mathbb{k}) \simeq M(n, \mathbb{k}), \quad \alpha \mapsto (\alpha(x_{i,j})), \quad \text{as (multiplicative) monoids}.$$

$$Det = \sum_{\sigma \in S_n} x_{1,\sigma(1)} \cdots x_{n,\sigma(n)} \in G(B),$$
$A = B[\text{Det}^{-1}]$ is a Hopf algebra, $\text{Alg}_k(A, k)$ is a group under convolution, and

$$\text{Alg}_k(A, k) \simeq GL(n, k)$$

as groups.

An affine algebraic group is a pair $(G, A)$, where $G$ is a group, $A$ is a certain commutative Hopf algebra over $k$, and $G \simeq \text{Alg}_k(A, k)$. We write $A = A(G)$. $A$ determines $(G, A)$. 
5. The General Theory of Hopf Algebras Begins

With Sweedler’s book [Swe 1969b] the study of Hopf algebras in general was just underway. Previously Hopf algebras of interest were either cocommutative or commutative. $kG$, $U(L)$ are cocommutative, $A(G)$ is commutative. $U(L)$ and $A(G)$ are precursors of quantum groups.

Efforts were made to prove known results, or discover new ones, for affine groups using Hopf algebra methods, see [Swe 1969a], [Sul 1971], [Sul 1973], [Tak 1972a], [Tak 1972b]. The study of cocommutative Hopf algebras was pursued [Swe 1967], [New–Swe 1979].

Connections were made with many aspects of algebra. Hopf algebras were seen as rings which were interesting in their own right. There was an effort to generalize results about the group
algebras of finite groups to f-d Hopf algebras. For these generalizations would hold for both these group algebras and restricted enveloping algebras.

Hopf algebras were constructed as vector spaces on certain diagrams which can be combined (which gives rise to a product) and decomposed (which gives rise to a coproduct). See [Gross–Lar 1989], [Connes–Krei 2001].

The antipode was scrutinized since it is such an important part of the structure of a Hopf algebra. In [Tak 1971] a Hopf algebra is given where $S$ is not bijective. If $A$ is commutative or cocommutative $S^2 = \text{Id}_A$ and therefore $S$ is bijective.

In [Taft 1971] f-d examples $T_n$, where $n \geq 1$, are given where $S^2$ has finite order $n$. $T_2$ is Sweedler’s example. $\text{Dim } T_n = n^2$. 
Example 3 Let \( n \geq 1 \) and suppose \( q \in k \) is a primitive \( n^{th} \) root of unity. \( T_n \) is generated as an algebra by \( a, x \) subject to the relations

\[
xa = qax, \quad x^n = 0, \quad \text{and} \quad a^n = 1
\]

and the coalgebra structure is determined by

\[
\Delta(a) = a \otimes a \quad \text{and} \quad \Delta(x) = 1 \otimes x + x \otimes a. \quad (2)
\]

Necessarily \( \epsilon(a) = 1 \), thus \( A \in G(T_n) \), and \( \epsilon(x) = 0 \). An \( x \in A \) such that (2) is satisfied for some \( a \in G(A) \) is skew-primitive. Compare with (1). When \( a = 1 \) note \( x \) is primitive. The boxed expressions are basic types of defining relations for the quantized enveloping algebras (here \( q \) is not a root of unity).

In 1975 Kaplansky postulated 10 conjectures [Kap 1975] about Hopf algebras some of which are open. They have focused the research of many.
6. Some Fundamental Results for F-D Hopf Algebras

$G$ is a finite group. $\Lambda = \sum_{h \in G} h$ satisfies

$$g\Lambda = \sum_{h \in G} gh = \Lambda = 1_k\Lambda = \epsilon(g)\Lambda$$

for $g \in G$ and

$$\epsilon(\Lambda) = \sum_{h \in G} \epsilon(h) = \sum_{h \in G} 1_k = |G|1_k.$$ 

Maschke’s Theorem can be formulated: All left $kG$-modules are completely reducible if and only if $\epsilon(\Lambda) \neq 0$.

$\Lambda \in A$ is a left (resp. right) integral for $A$ if $a\Lambda = \epsilon(a)\Lambda$ (resp. $\Lambda a = \epsilon(a)\Lambda$) for all $a \in A$. There is non-zero (left) integral $\Lambda$ for $A$ iff $A$ is f-d [Swe 1969c] in which case any (left) integral for $A$ is a scalar multiple of $\Lambda$. 
Now suppose $A$ is f-d. All left $A$-modules are completely reducible if and only if $\epsilon(\Lambda) \neq (0)$ [Lar–Swe 1969c].

There is a $g \in G(A)$ which relates left and right integrals for $A$ and an $\alpha \in G(A^*)$ which does the same for $A^*$. Let $A \xrightarrow{\sigma_g} A$, $A^* \xrightarrow{\sigma_\alpha} A^*$ denote conjugation by $g$, $\alpha$ respectively. Then $\sigma_g, \sigma_\alpha^*$ commute and

$$S^4 = \sigma_g \circ \sigma_\alpha^*.$$

Thus $S$ has finite order. See [Rad 1976].

There is a Hopf algebra analog of Lagrange’s Theorem for a finite group $G$. Let $H$ be a subgroup of $G$. Then $|H|$ divides $|G|$ if and only if $\mathbb{k}G$ is a free left $\mathbb{k}H$-module. A most sought after result was finally established in [Nic–Zel 1989]:

**Theorem 1** A f-d Hopf algebra is a free (left) module over its sub-Hopf algebras.
The proof, which is quite subtle, is based on the notion of relative Hopf module, a generalization of Hopf module.

Now let $A$ be any Hopf algebra over $\mathbb{k}$. Then a left $A$-Hopf module is a triple $(M, \mu, \rho)$, where $A \otimes M \xrightarrow{\mu} M$ is a left $A$-module, $M \xrightarrow{\rho} A \otimes M$ is a left $A$-comodule which satisfy a certain compatibility.

All left $A$-Hopf modules are free and have a special basis [Swe 1969c]. This result is one of the most important in the theory of Hopf algebras. In particular it accounts for basic results about integrals.

Suppose $A$ is semisimple (as an algebra). Then $A$ is f-d [Swe 1969c]. If the characteristic of $\mathbb{k}$ is 0 then $A^*$ is also semisimple and $S^2 = \text{Id}_A$ [Lar–Rad 1988a, 1988b]. If the characteristic is positive and $A$, $A^*$ are semisimple $S^2 = \text{Id}_A$ [Eting–Gel 1998].
7. The Advent of Quantum Groups - An Explosion of Activity

Drinfel’d’s paper [Drinfel’d 2007] presented at the ICM held at Berkeley, CA, in 1986 described new classes of non-commutative, non-commutative Hopf algebras, which we refer to as quantum groups, derived from commutative or cocommutative ones through ”quantization”. His paper pointed to connections of quantum groups with physics, algebra, non-commutative geometry, representation theory, and topology.

There are general text books on quantum groups. These include [Char–Press 1994], [Kassel 1995], and [Majid 1995].

Some important consequences for Hopf algebras were the introduction of the quantized enveloping algebras, of quasitriangular Hopf algebras, an important example of which is the Drinfel’d double, and later introduction of the small quantum groups of Lusztig. The paper [Majid 1990] is a good entry point for Hopf algebraists to make first foray into quantum groups.

There was a flurry of activity to find quantizations of Hopf algebras associated with certain affine groups. Sometime later quasitriangular Hopf algebras were seen to account for regular isotopy invariants of oriented knots and links in a very concrete manner [Kauff–Rad 2001].
For us a quasitriangular Hopf algebra over \( k \) is a pair \((A, R)\), where \( A \) is a Hopf algebra over \( k \), and \( R \in A \otimes A \) satisfies certain axioms which guarantee that it satisfies algebraists’ Yang–Baxter equation. When \( A \) is f-d the Drinfel’d double \((D(A), R)\) can be constructed. Both \( A, A^{* \cop} \) are subHopf algebras of \( D(A) \) and multiplication \( A^{*} \otimes A \rightarrow D(A) \) is a linear isomorphism.

Thus f-d quasitriangular Hopf algebras abound. The invariants they produce are almost a total mystery. Concerning the double, there is a rather mysterious connection comes to light in [Kauff–Rad 1993] between the formula for \( S^4 \) and when a certain 3-manifold invariant arises from \( D(A), R \). The invariant was first described in [Henn 1996].

When \( A \) is f-d the category of left \( D(A) \)-modules is equivalent to the Yetter-Drinfel’d category
Objects are triples $\left( M, \mu, \rho \right)$, where $A \otimes M \xrightarrow{\mu} M$ and $M \xrightarrow{\rho} A \otimes M$ are left $A$-module and left $A$-comodule structures on $M$ respectively satisfying a rather complicated compatibility condition reflecting the commutation relation for multiplication in $D(A)$.

This condition is quite different from the Hopf module compatibility condition. Certain Hopf algebras in this category are important for the classification of f-d Hopf algebras when $A = \mathbb{k}G$ is the group algebra of a finite abelian group.
8. Classification of Pointed Hopf Algebras

Let $A$ be any Hopf algebra. $A_0$ denotes the sum of all the simple subcoalgebras of $A$ and $A$ is pointed if these are 1-dimensional. In this case $A_0 = \mathbb{k}G(A)$ and is a subHopf algebra of $A$. The quantized enveloping algebras, and the small quantum groups of Lusztig, are pointed.

Suppose $A_0$ is a subHopf algebra of $A$. There is a graded pointed irreducible Hopf algebra $\text{gr}(A)$ with $\text{gr}(A)_0 = \text{gr}(A)(0) = A_0$. We now outline the strategy of [Andrus–Schn 2002] for determining the structure of $A$.

Let $\text{gr}(A) \xrightarrow{\pi} A_0$ be the projection. The right covariants $R = \text{gr}(A)^{co\pi}$ form a graded pointed irreducible Hopf algebra in the category $A_0 \mathcal{YD}$ and there is an isomorphism of $\text{gr}(A) \simeq R \times A_0$ with a canonical biproduct [Rad 1985].
For a discussion of Hopf algebras in $A_0\mathcal{YD}$ and related categories see [Majid 1992]. We have $R(0) = k1$ and $R(1) = P(R)$. The Nichols algebra associated with $V = P(R)$ is $B(V)$, the subalgebra of $R$ generated by $V$. We note $B(V)$ is analogous the enveloping algebra of a Lie algebra. Steps for classification of $A$:

(1) Determine the structure of $B(V)$;

(2) Determine all Hopf algebras $B$ over $k$ such that $\text{gr}(B) \cong B(V) \times A_0$;

(3) Determine whether or not $B(V) = R$ (in which case $A = B$ for some $B$ of (2)).

Let $B$ be a Hopf algebra over $k$. Then for any object $V$ of $B\mathcal{YD}$ there is a graded pointed irreducible Hopf algebra $B(V)$ in $B\mathcal{YD}$ which
is determined by $\mathcal{B}(V)(1) = V$ and $V$ generates $\mathcal{B}(V)$ as an algebra. These are the Nichols algebras. They have been described in many ways in important cases which have been studied in [Lusztig 1993], [Rosso 1995, 1998], [Heck 2004]. Basic results about them are nontrivial.

Andruskiewitsch and Schneider have used them in classifying f-d pointed Hopf algebras when $\mathbb{k}$ is algebraically closed of characteristic 0 and $G(A)$ is commutative with mild restrictions on $|G(A)|$ [Andrus–Schn 2010]. The similarities between these Hopf algebras and Lustig’s small quantum groups are striking.
References


