Hopf Algebras, an Overview

School of Mathematics and Computational Science

Sun Yat-sen University
Guangzhou, PRC
January 22, 2010

David E. Radford
The University of Illinois at Chicago
Chicago, IL, USA

0. Introduction

Hopf algebras are named after Heinz Hopf who discovered them the last century in the context of algebraic topology [**Hopf 1941**]. They arise in many areas of mathematics. Group algebras, enveloping algebras of Lie algebras, and quantum groups are examples of Hopf algebras. Certain Hopf algebras give rise to invariants of knots and links. We will describe Hopf algebras, discuss basic examples and fundamental results, and trace the development of the theory.

The Hopf algebras we discuss here are technically different from those coming from algebraic topology. See the very interesting discussion in [**Haz 2008**].

Objects are vector spaces over a field $\Bbbk$ and maps are $\Bbbk$-linear. $\otimes = \otimes_{\Bbbk}$. "f-d" = finite-dimensional.

1. A Basic Example and Definitions

G is a group and $A = \Bbbk G$ is the group algebra of G over $\Bbbk$. Let $g, h \in G$. The algebra structure:

$\Bbbk G \otimes \Bbbk G \xrightarrow{m} \Bbbk G$	$m(g \otimes h) = gh$
$k \xrightarrow{\eta} \Bbbk G$	$\eta(1_k) = e = 1_{\Bbbk G}$

The **co**algebra structure:

$\Bbbk G \xrightarrow{\Delta} \Bbbk G \otimes \Bbbk G$	$\Delta(g) = g \otimes g$
$\Bbbk G \xrightarrow{\epsilon} k$	$\epsilon(g) = 1_k$

The map which accounts for inverses:

$\Bbbk G \xrightarrow{S} \Bbbk G$	$S(g) = g^{-1}$

Observe that

$$\Delta(gh) = gh \otimes gh = (g \otimes g)(h \otimes h) = \Delta(g)\Delta(h),$$

$$\epsilon(gh) = 1_{\Bbbk} = 1_{\Bbbk}1_{\Bbbk} = \epsilon(g)\epsilon(h),$$

$$S(gh) = (gh)^{-1} = h^{-1}g^{-1} = S(h)S(g),$$

$$gS(g) = gg^{-1} = 1_{\Bbbk G} = 1_{\Bbbk}1_{\Bbbk G} = \epsilon(g)1_{\Bbbk G},$$

and

$$S(g)g = g^{-1}g = 1_{\Bbbk G} = \epsilon(g)1_{\Bbbk G}.$$

In particular $\boxed{\Delta, \epsilon \text{ are algebra maps}}$ and S is determined by

$$\boxed{gS(g) = \epsilon(g)1_{\Bbbk G} = S(g)g.}$$

We generalize the system $(\Bbbk G, m, \eta, \Delta, \epsilon, S)$.

A **Hopf algebra over** $\mathbb{k}$ is a tuple $(A, m, \eta, \Delta, \epsilon, S)$, where (A, m, η) is an algebra over $\mathbb{k}$:

$A \otimes A \xrightarrow{m} A$	$m(a \otimes b) = ab$
$k \xrightarrow{\eta} A$	$\eta(1_k) = 1_A$

(A, Δ, ϵ) is a **coalgebra over** $\mathbb{k}$:

$A \xrightarrow{\Delta} A \otimes A$	$\Delta(a) = a_{(1)} \otimes a_{(2)}$
$A \xrightarrow{\epsilon} k$	

and $\boxed{A \xrightarrow{S} A}$ is an "antipode" where certain axioms are satisfied.

Comments: $\Delta(a) \in A \otimes A$ is usually a *sum* of tensors; thus $\Delta(a) = a_{(1)} \otimes a_{(2)}$ **is a notation**, called the Heyneman-Sweedler notation. Δ is called the **coproduct** and ϵ the **counit**.

The axioms for a Hopf algebra:

(A, m, η) is an (associative) algebra:

$$\boxed{(ab)c = a(bc), \qquad\qquad 1a = a = a1}$$

(A, Δ, ϵ) is a (coassociative) coalgebra:

$$\boxed{\begin{array}{l} a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)}, \\[1em] \epsilon(a_{(1)})a_{(2)} = a = a_{(1)}\epsilon(a_{(2)}) \end{array}}$$

Δ is an algebra map:

$$\boxed{\Delta(ab) = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}, \qquad \Delta(1) = 1 \otimes 1,}$$

ϵ is an algebra map:

$$\boxed{\epsilon(ab) = \epsilon(a)\epsilon(b), \qquad \epsilon(1) = 1,} \qquad \text{and}$$

$$\boxed{a_{(1)}S(a_{(2)}) = \epsilon(a)1 = S(a_{(1)})a_{(2)})}$$

for all $a, b \in A$. From now on A denotes a Hopf algebra over $\mathbb{k}$.

2. Basic Properties and More Definitions

1. A has a unique antipode and

$\boxed{S(ab) = S(b)S(a), \qquad S(1) = 1.}$ Also

$$\boxed{\Delta(S(a)) = S(a_{(2)}) \otimes S(a_{(1)}), \quad \epsilon(S(a)) = \epsilon(a).}$$

2. $a \in A$ *is cocommutative* if $a_{(1)} \otimes a_{(2)} = a_{(2)} \otimes a_{(1)}$; A *is cocommutative* if all $a \in A$ are. $\Bbbk G$ is cocommutative. A *is commutative* if $ab = ba$ for all $a, b \in A$. $\Bbbk G$ is commutative iff G is.

3. $a \in A$ *is grouplike* if $\Delta(a) = a \otimes a$ and $\epsilon(a) = 1$. $1 \in A$ is grouplike. The set $G(A)$ of grouplike elements of A is linearly independent (coalgebra fact). $G(\Bbbk G) = G$.

4. $G(A)$ is a group under multiplication and $S(g) = g^{-1}$ for $g \in G(A)$. Thus if A is f-d then $G(A)$ is a finite group.

5. Let M, N be left A-modules (regard A as an algebra). Then $M \otimes N$ is a left A-module where $\boxed{a \cdot (m \otimes n) = a_{(1)} \cdot m \otimes a_{(2)} \cdot n.}$ For $g \in G \subseteq \Bbbk G$ note $g \cdot (m \otimes n) = g \cdot m \otimes g \cdot n$.

6. If S is bijective then $(A, m^{op}, \eta, \Delta, \epsilon, S^{-1})$, $(A, m, \eta, \Delta^{cop}, \epsilon, S^{-1})$ are Hopf algebras, where $m^{op}(a \otimes b) = ba$ and $\Delta^{cop}(a) = a_{(2)} \otimes a_{(1)}$. $(A, m^{op}, \eta, \Delta^{cop}, \epsilon, S)$ is a Hopf algebra.

7. If A is f-d then S is bijective.

8. If $(A, m, \eta, \Delta, \epsilon, S)$ is f-d $(A^*, \Delta^*, \epsilon^*, m^*, \eta^*, S^*)$ is a f-d Hopf algebra over $\Bbbk$.

9. Let C be a (coassociative) coalgebra. Every f-d subspace of C generates a f-d subcoalgebra. Thus A has simple subcoalgebras, and all are f-d. $\Bbbk 1$, more generally $\Bbbk g$ for $g \in G(A)$, is a simple subcoalgebra of A.

The reader is referred to any basic text on Hopf algebras: **[Swe 1967]**, **[Abe 1980]**, **[Mont 1993]**, **[D–N–Rai 2001]**.

3. The Enveloping Algebra

L is a Lie algebra over $\mathbb{k}$. The enveloping algebra $U(L)$ is a cocommutative Hopf algebra over $\mathbb{k}$ where

$$\boxed{\Delta(\ell) = 1 \otimes \ell + \ell \otimes 1,} \tag{1}$$

$\epsilon(\ell) = 0$, and $S(\ell) = -\ell$ for $\ell \in L$. An $\ell \in A$ such that (1) holds is *primitive*. The set of primitives $P(A)$ of A is a subspace and a a Lie algebra under associative bracket. $A = U(L)$ is *pointed irreducible* meaning $\mathbb{k}1$ is the only simple subcoalgebra of A.

Assume $\mathbb{k}$ has characteristic 0. Then $P(U(L))) = L$ and the Milnor-Moore Theorem [**Mil−Mo 1965**] characterizes the finitely generated graded pointed irreducible Hopf algebras over $\mathbb{k}$ as the enveloping algebras of f-d Lie algebras over $\mathbb{k}$. See [**Haz 2008**] also.

4. Affine Algebraic Groups

Coalgebras C over $\mathbb{k}$ provide many examples of algebras. The linear dual $C^* = \mathrm{Hom}_{\mathbb{k}}(C, \mathbb{k})$ is an algebra over $\mathbb{k}$ with *convolution product*:

$$\boxed{1_{C^*} = \epsilon \qquad \text{and} \qquad ab(c) = a(c_{(1)})b(c_{(2)})}$$

for all $a, b \in C^*$ and $c \in C$.

Example 1 *C has basis $c_0, c_1, c_2, \ldots$ and*

$$\boxed{\Delta(c_n) = \textstyle\sum_{\ell=0}^{n} c_{n-\ell} \otimes c_\ell, \qquad \epsilon(c_n) = \delta_{n,0}.}$$

As $(ab)(c_n) = \displaystyle\sum_{\ell=0}^{n} a(c_{n-\ell})b(c_\ell)$*,*

$$C^* \simeq \mathbb{k}[[x]], \qquad a \mapsto \sum_{n=0}^{\infty} a(c_n)x^n.$$

Example 1 suggests coalgebra connections with combinatorics [**Rom – Rota 1978**], [**Rota 1978**].

Example 2 *Let $n \geq 1$ and $C(n, \Bbbk)$ have basis $x_{i,j}$, $1 \leq i, j \leq n$ and*

$$\boxed{\Delta(x_{i,j}) = \textstyle\sum_{\ell=1}^{n} x_{i,\ell} \otimes x_{\ell,j}, \qquad \epsilon(x_{i,j}) = \delta_{i,j}.}$$

As $(ab)(x_{i,j}) = \displaystyle\sum_{\ell=1}^{n} a(x_{i,\ell})b(x_{\ell,j})$,

$C^* \simeq M(n, \Bbbk), \quad a \mapsto (a_{i,j})$, *where* $a_{i,j} = a(x_{i,j})$.

We now continue. $B = S(C(n, \Bbbk))$ is the free commutative $\Bbbk$-algebra on $C(n, \Bbbk)$. Now let $\Delta : B \longrightarrow B \otimes B$ and $\epsilon : B \longrightarrow \Bbbk$ be the algebra maps determined on $x_{i,j}$ as in Example 2. $Alg_{\Bbbk}(B, \Bbbk)$ is closed under the convolution product, contains ϵ, and

$$Alg_{\Bbbk}(B, \Bbbk) \simeq M(n, \Bbbk), \qquad \alpha \mapsto (\alpha(x_{i,j})),$$

as (multiplicative) monoids.

$$\mathrm{Det} = \sum_{\sigma \in S_n} x_{1,\sigma(1)} \cdots x_{n,\sigma(n)} \in G(B),$$

$A = B[\mathrm{Det}^{-1}]$ is a Hopf algebra, $Alg_{\mathbb{k}}(A, \mathbb{k})$ is a group under convolution, and

$$\boxed{Alg_{\mathbb{k}}(A, \mathbb{k}) \simeq GL(n, \mathbb{k}) \qquad \text{as groups.}}$$

An affine algebraic group is a pair (G, A), where G is a group, A is a certain commutative Hopf algebra over $\mathbb{k}$, and $G \simeq Alg_{\mathbb{k}}(A, \mathbb{k})$. We write $A = A(G)$. A determines (G, A).

5. The General Theory of Hopf Algebras Begins

With Sweedler's book [**Swe 1969b**] the study of Hopf algebras in general was just underway. Previously Hopf algebras of interest were either cocommutative or commutative. $\Bbbk G$, $U(L)$ are cocommutative, $A(G)$ is commutative. $U(L)$ and $A(G)$ are precursors of quantum groups.

Efforts were made to prove know results, or discover new ones, for affine groups using Hopf algebra methods, see [**Swe 1969a**], [**Sul 1971**], [**Sul 1973**], [**Tak 1972a**], [**Tak 1972b**]. The study of cocommutative Hopf algebras was pursued [**Swe 1967**], [**New–Swe 1979**].

Connections were made with many aspects of algebra. Hopf algebras were seen as rings which were interesting in their own right. There was an effort to generalize results about the group algebras of finite groups to f-d Hopf algebras. For these generalizations would hold for both these group algebras and restricted enveloping algebras.

Hopf algebras were constructed as vector spaces on certian diagrams which can be combined (which gives rise to a product) and decomposed (which gives rise to a coproduct). See **[Gross–Lar 1989]**, **[Connes–Krei 2001]**.

The antipode was scrutinized since it is such an important part of the structure of a Hopf algebra. In **[Tak 1971]** a Hopf algebra is given where S is not bijective. If A is commutative or cocommutative $S^2 = \mathrm{Id}_A$ and therefore S is bijective.

In **[Taft 1971]** f-d examples T_n, where $n \geq 1$, are given where S^2 has finite order n. T_2 is Sweedler's example. Dim $T_n = n^2$.

Example 3 *Let $n \geq 1$ and suppose $q \in \mathbb{k}$ is a primitive n^{th} root of unity. T_n is generated as an algebra by a, x subject to the relations*

$$\boxed{xa = qax}, \quad x^n = 0, \quad \textit{and} \quad a^n = 1$$

and the coalgebra structure is determined by

$$\boxed{\Delta(a) = a \otimes a \textit{ and } \Delta(x) = 1 \otimes x + x \otimes a.} \quad (2)$$

Necessarily $\epsilon(a) = 1$, thus $A \in G(T_n)$, and $\epsilon(x) = 0$. An $x \in A$ such that (2) is satisfied for some $a \in G(A)$ is skew-primitive. Compare with (1). When $a = 1$ note x is primitive. The boxed expressions are basic types of defining relations for the quantized enveloping algebras (here q is not a root of unity).

In 1975 Kaplansky postulated 10 conjectures [**Kap 1975**] about Hopf algebras some of which are open. They have focused the research of many.

6. Some Fundamental Results for F-D Hopf Algebras

G is a finite group. $\Lambda = \sum_{h \in G} h$ satisfies

$$g\Lambda = \sum_{h \in G} gh = \Lambda = 1_{\Bbbk}\Lambda = \epsilon(g)\Lambda$$

for $g \in G$ and

$$\epsilon(\Lambda) = \sum_{h \in G} \epsilon(h) = \sum_{h \in G} 1_{\Bbbk} = |G|1_{\Bbbk}.$$

Maschke's Theorem can be formulated: All left $\Bbbk G$-modules are completely reducible if and only if $\epsilon(\Lambda) \neq 0$.

$\Lambda \in A$ is a *left (*resp. *right) integral for* A if $\boxed{a\Lambda = \epsilon(a)\Lambda}$ (resp. $\boxed{\Lambda a = \epsilon(a)\Lambda}$) for all $a \in A$. There is non-zero (left) integral Λ for A iff A is f-d **[Swe 1969c]** in which case any (left) integral for A is a scalar multiple of Λ.

Now suppose A is f-d. All left A-modules are completely reducible if and only if $\epsilon(\Lambda) \neq (0)$ [**Lar–Swe 1969c**].

There is a $g \in G(A)$ which relates left and right integrals for A and an $\alpha \in G(A^*)$ which does the same for A^*. Let $A \xrightarrow{\sigma_g} A$, $A^* \xrightarrow{\sigma_\alpha} A^*$ denote conjugation by g, α respectively. Then $\sigma_g, \sigma_\alpha^*$ commute and

$$\boxed{S^4 = \sigma_g \circ \sigma_\alpha^*}.$$

Thus S has finite order. See [**Rad 1976**].

There is a Hopf algebra analog of Lagrange's Theorem for a finite group G. Let H be a subgroup of G. Then $|H|$ divides $|G|$ if and only if $\Bbbk G$ is a free left $\Bbbk H$-module. A most sought after result was finally established in [**Nic–Zel 1989**]:

Theorem 1 *A f-d Hopf algebra is a free (left) module over its sub-Hopf algebras.*

The proof, which is quite subtle, is based on the notion of relative Hopf module, a generalization of Hopf module.

Now let A be any Hopf algebra over $\mathbb{k}$. Then a left A-Hopf module is a triple (M, μ, ρ), where $A \otimes M \xrightarrow{\mu} M$ is a left A-module, $M \xrightarrow{\rho} A \otimes M$ is a left A-comodule which satisfy a certain compatibility.

All left A-Hopf modules are free and have a special basis **[Swe 1969c]**. This result is one of the most important in the theory of Hopf algebras. In particular it accounts for basic results about integrals.

Suppose A is semisimple (as an algebra). Then A is f-d **[Swe 1969c]**. If the characteristic of $\mathbb{k}$ is 0 then A^* is also semisimple and $S^2 = \mathrm{Id}_A$ **[Lar–Rad 1988a, 1988b]**. If the characteristic is positive and A, A^* are semisimple $S^2 = \mathrm{Id}_A$ **[Eting–Gel 1998]**.

7. The Advent of Quantum Groups - An Explosion of Activity

Drinfel'd's paper [**Drinfel′d 2007**] presented at the ICM held at Berkeley, CA, in 1986 described new classes of non-commutative, non-commutative Hopf algebras, which we refer to as quantum groups, derived from commutative or cocommutative ones through "quantization". His paper pointed to connections of quantum groups with physics, algebra, non-commutative geometry, representation theory, and topology.

For connections with representation theory see [**Lusztig 1993**] and [**Char−Press 1994**]. For connections with non-commutative geometry see [**Manin 1988, 1991**], and for connections with invariants of knots, links, and 3-manifolds see [**Kassel−Rosso−Tur1997**].

There are general text books on quantum groups. These include [**Char−Press 1994**], [**Kassel 1995**], and [**Majid 1995**].

Some important consequences for Hopf algebras were the introduction of the quantized enveloping algebras, of quasitriangular Hopf algebras, an important example of which is the Drinfel'd double, and later introduction of the small quantum groups of Lusztig. The paper [**Majid 1990**] is a good entry point for Hopf algebraists to make first foray into quantum groups.

There was a flurry of activity to find quantizations of Hopf algebras associated with certain affine groups. Sometime later quasitriangular Hopf algebras were seen to account for regular isotopy invariants of oriented knots and links in a very concrete manner [**Kauff−Rad 2001**].

For us a *quasitriangular Hopf algebra over* $\mathbb{k}$ is a pair (A, R), where A is a Hopf algebra over $\mathbb{k}$, and $R \in A \otimes A$ satisfies certain axioms which guarantee that it satisfies algebraists' Yang–Baxter equation. When A is f-d the Drinfel'd double $(D(A), R)$ can be constructed. Both $A, A^{*\,cop}$ are subHopf algebras of $D(A)$ and multiplication $A^* \otimes A \longrightarrow D(A)$ is a linear isomorphism.

Thus f-d quasitriangular Hopf algebras abound. The invariants they produce are almost a total mystery. Concerning the double, there is a rather mysterious connection comes to light in **[Kauff–Rad 1993]** between the formula for S^4 and when a certain 3-manifold invariant arises from $D(A), R)$. The invariant was first described in **[Henn 1996]**.

When A is f-d the category of left $D(A)$-modules is equivalent to the Yetter-Drinfel'd category

${}_A\mathcal{YD}$ **[Yetter 1990]**, **[Majid 1991]**. Objects are triples (M, μ, ρ), where $A \otimes M \xrightarrow{\mu} M$ and $M \xrightarrow{\rho} A \otimes M$ are left A-module and left A-comodule structures on M respectively satisfying a rather complicated compatibility condition reflecting the commutation relation for multiplication in $D(A)$.

This condition is quite different from the Hopf module compatibility condition. Certain Hopf algebras in this category are important for the classification of f-d Hopf algebras when $A = \Bbbk G$ is the group algebra of a finite abelian group.

8. Classification of Pointed Hopf Algebras

Let A be any Hopf algebra. A_0 denotes the sum of all the simple subcoalgebras of A and A *is pointed* if these are 1-dimensional. In this case $A_0 = \Bbbk G(A)$ and is a subHopf algebra of A. The quantized enveloping algebras, and the small quantum groups of Lusztig, are pointed.

Suppose A_0 is a subHopf algebra of A. There is a graded pointed irreducible Hopf algebra $\mathrm{gr}(A)$ with $\mathrm{gr}(A)_0 = \mathrm{gr}(A)(0) = A_0$. We now outline the strategy of [**Andrus–Schn 2002**] for determining the structure of A.

Let $\mathrm{gr}(A) \xrightarrow{\pi} A_0$ be the projection. The right covariants $R = \mathrm{gr}(A)^{co\,\pi}$ form a graded pointed irreducible Hopf algebra **in the category** ${}_{A_0}\mathcal{YD}$ and there is an isomorphism of $\mathrm{gr}(A) \simeq R \times A_0$ with a canonical biproduct [**Rad 1985**].

For a discussion of Hopf algebras in ${}_{A_0}\mathcal{YD}$ and related categories see [**Majid 1992**]. We have $R(0) = \Bbbk 1$ and $R(1) = P(R)$. The Nichols algebra associated with $V = P(R)$ is $\mathcal{B}(V)$, the subalgebra of R generated by V. We note $\mathcal{B}(V)$ is analogous the enveloping algebra of a Lie algebra. Steps for classification of A:

(1) Determine the structure of $\mathcal{B}(V)$;

(2) Determine all Hopf algebras B over $\Bbbk$ such that $\mathrm{gr}(B) \simeq \mathcal{B}(V) \times A_0$;

(3) Determine whether or not $\mathcal{B}(V) = R$ (in which case $A = B$ for some B of (2)).

Let B be a Hopf algebra over $\Bbbk$. Then for any object V of ${}_B\mathcal{YD}$ there is a graded pointed irreducible Hopf algebra $\mathcal{B}(V)$ in ${}_B\mathcal{YD}$ which is determined by $\mathcal{B}(V)(1) = V$ and V generates $\mathcal{B}(V)$ as an algebra. These are the Nichols algebras. They have been described in many ways in important cases which have been studied in [**Lusztig 1993**], [**Rosso 1995, 1998**], [**Heck 2004**]. Basic results about them are nontrivial.

Andruskiewitsch and Schneider have used them in classifying f-d pointed Hopf algebras when $\Bbbk$ is algebraically closed of characteristic 0 and $G(A)$ is commutative with mild restrictions on $|G(A)|$ [**Andrus–Schn 2010**]. The similarities between these Hopf algebras and Lustig's small quantum groups are striking.

References

[**Abe 1980**] Abe, Eiichi. Hopf algebras. Cambridge Tracts in Mathematics, 74. Cambridge University Press, Cambridge-New York, 1980. xii+284 pp.

[**Andrus-Fan-Gra-Ven 2010**] N. Andruskiewitsch, F. Fantino, M. Graa, L. Vendramin. Pointed Hopf algebras over the sporadic groups. math.QA arXiv:1001.1108

[**Andrus-Nat 2001**] Andruskiewitsch, Nicols; Natale, Sonia. Counting arguments for Hopf algebras of low dimension. Tsukuba J. Math. 25 (2001), no. 1, 187–201.

[**Andrus-Schn 2002**] Andruskiewitsch, Nicols; Schneider, Hans-Jürgen. Pointed Hopf algebras. New directions in Hopf algebras, 1–68, Math. Sci. Res. Inst. Publ., 43, Cambridge Univ. Press, Cambridge, 2002.

[**Andrus-Schn 2010**] N. Andruskiewitsch; Hans-Jürgen Schneider. On the classification of finite-dimensional pointed Hopf algebras. math.QA/0502157

[**Char-Press 1994**] Chari, Vyjayanthi; Pressley, Andrew. A guide to quantum groups. Cambridge University Press, Cambridge, 1994. xvi+651 pp.

[**Bea-Dasc 2004**] Beattie, M.; Dascalescu, S. Hopf algebras of dimension 14. J. London Math. Soc. (2) 69 (2004), no. 1, 65–78.

[**Connes-Krei 2001**] Connes, A.; Kreimer, D. From local perturbation theory to Hopf- and Lie-algebras of Feynman graphs. Mathematical physics in mathematics and physics (Siena, 2000), 105–114, Fields Inst. Commun., 30, Amer. Math. Soc., Providence, RI, 2001.

[**D-N-Rai 2001**] Dascalescu, Sorin; Nastasescu, Constantin; Raianu, Serban. Hopf algebras. An introduction. Monographs and Textbooks in Pure and Applied Mathematics, 235. Marcel Dekker, Inc., New York, 2001. x+401 pp.

[**Drinfel'd 1987**] Drinfel'd, V. G. Quantum groups. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 798–820, Amer. Math. Soc., Providence, RI, 1987.

[**Gross-Lar 1989**] Grossman, Robert; Larson, Richard G. Hopf-algebraic structure of families of trees. J. Algebra 126 (1989), no. 1, 184–210.

[**Haz 2008**] M. Hazewinkel. Niceness Theorems (2008) arXiv:0810.5691

[**Heck 2004**] I. Heckenberger. Finite dimensional rank 2 Nichols algebras of diagonal type II: Classification. math.QA/0404008

[**Heck-Sch 2008**] I. Heckenberger; H. -J. Schneider. Root systems and Weyl groupoids for Nichols algebras. (2008) math.QA arXiv:0807.0691

[**Henn 1996**] Hennings, Mark. Invariants of links and 3-manifolds obtained from Hopf algebras. J. London Math. Soc. (2) 54 (1996), no. 3, 594–624.

[**Hey-Swe 1969**] Heyneman, Robert G.; Sweedler, Moss E. Affine Hopf algebras. II. J. J. Algebra 13 1969 192–241.

[**Hey-Swe 1970**] Heyneman, Robert G.; Sweedler, Moss E. Affine Hopf algebras. II. J. Algebra 16 1970 271–297.

[**Hopf 1941**] Hopf, Heinz. Über die Topologie der Gruppen- Mannigfaltigkeiten und ihre Verallgemeinerungen. Ann. of Math. (2) 42, (1941). 22–52.

[**Joni-Rota 197**] Joni, S. A.; Rota, G.-C. Coalgebras and bialgebras in combinatorics. Stud. Appl. Math. 61 (1979), no. 2, 93–139.

[**Kap 1975**] Kaplansky, Irving. Bialgebras. Lecture Notes in Mathematics. Department of Mathematics, University of Chicago, Chicago, Ill., 1975. iv+57 pp.

[**Kas-Som-Zhu 2006**] Kashina, Yevgenia; Sommerhuser, Yorck; Zhu, Yongchang. On higher Frobenius-Schur indicators. Mem. Amer. Math. Soc. 181 (2006), no. 855, viii+65 pp.

[**Kassel 1995**] Kassel, Christian. Quantum groups. Graduate Texts in Mathematics, 155. Springer-Verlag, New York, 1995. xii+531 pp.

[**Kassel-Rosso-Tur 1997**] Kassel, Christian; Rosso, Marc; Turaev, Vladimir. Quantum groups and knot invariants. Panoramas et Synthses Panoramas and Syntheses, 5. Socit Mathmatique de France, Paris, 1997. vi+115 pp.

[**Kauff-Rad 2001**] Kauffman, Louis H.; Radford, David E. Oriented quantum algebras and invariants of knots and links. J. Algebra 246 (2001), no. 1, 253–291.

[**Kauff-Rad 1993**] Kauffman, Louis H.; Radford, David E. A necessary and sufficient condition for a finite-dimensional Drinfel?d double to be a ribbon Hopf algebra. J. Algebra 159 (1993), no. 1, 98–114.

[**Lar-Rad 1988a**] Larson, Richard G.; Radford, David E. Finite-dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple. J. Algebra 117 (1988), no. 2, 267–289.

[**Lar-Rad 1988b**] Larson, Richard G.; Radford, David E. Semisimple cosemisimple Hopf algebras. Amer. J. Math. 110 (1988), no. 1, 187–195.

[**Lar-Swe 1969**] Larson, Richard Gustavus; Sweedler, Moss Eisenberg. An associative orthogonal bilinear form for Hopf algebras. Amer. J. Math. 91 1969 75–94.

[**Lusztig 1993**] Lusztig, George. Introduction to quantum groups. Progress in Mathematics, 110. Birkhuser Boston, Inc., Boston, MA, 1993. xii+341 pp.

[**Majid 1990**] S. Majid. Physics for algebraists: non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction, J. Algebra 129 (1990), 1-91.

[**Majid 1991**] S. Majid. Doubles of quasitriangular Hopf algebras, Comm. Algebra 19 (1991), 3061-3073.

[**Majid 1992**] Majid, Shahn. Algebras and Hopf algebras in braided categories. (English summary) Advances in Hopf algebras (Chicago, IL, 1992), 55–105, Lecture Notes in Pure and Appl. Math., 158, Dekker, New York, 1994.

[**Majid 1995**] Majid, Shahn. Foundations of quantum group theory. (English summary) Cambridge University Press, Cambridge, 1995. x+607 pp.

[**Manin 1988**] Manin, Yu. I. Quantum groups and noncommutative geometry. Universit de Montral, Centre de Recherches Mathmatiques, Montreal, QC, 1988. vi+91 pp.

[**Manin 1991**] Manin, Yuri I. Topics in noncommutative geometry. M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1991. viii+164 pp.

[**Mas 1995a**] Masuoka, Akira. Self-dual Hopf algebras of dimension p^3 obtained by extension. J. Algebra 178 (1995), no. 3, 791–806.

[**Mas 1995b**] Masuoka, Akira. Semisimple Hopf algebras of dimension $6, 8$. Israel J. Math. 92 (1995), no. 1-3, 361–373.

[**Mas 1995c**] Masuoka, Akira. Semisimple Hopf algebras of dimension $2p$. Comm. Algebra 23 (1995), no. 5, 1931–1940.

[**Mil-Mo 1965**] Milnor, John W and John C Moore. On the structure of Hopf algebras, Ann. of Math. 81 (1965), 211-264.

[**Mont 1993**] Montgomery, Susan. Hopf algebras and their actions on rings. CBMS Regional Conference Series in Mathematics, 82. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993. xiv+238 pp.

[**Mont 2001**] Montgomery, Susan. Representation theory of semisimple Hopf algebras. Algebra—representation theory (Constanta, 2000), 189–218, NATO Sci. Ser. II Math. Phys. Chem., 28, Kluwer Acad. Publ., Dordrecht, 2001.

[**New-Swe 1979**] Newman, Kenneth; Sweedler, Moss E. A realization of the additive Witt group. Proc. Amer. Math. Soc. 76 (1979), no. 1, 39–42.

[**Nic 1978**] Nichols, Warren D. Bialgebras of type one. Comm. Algebra 6 (1978), no. 15, 1521–1552.

[**Nic-Zel 1989**] Nichols, Warren D. Zoeller, M. Bettina. A Hopf algebra freeness theorem. Amer. J. Math. 111 (1989), no. 2, 381–385.

[**Ng 2005**] Ng, Siu-Hung. Hopf algebras of dimension $2p$. Proc. Amer. Math. Soc. 133 (2005), no. 8, 2237–2242

[**Ng 2008**] Ng, Siu-Hung. Hopf algebras of dimension pq. II. J. Algebra 319 (2008), no. 7, 2772–2788.

[**Rad 1976**] Radford, David E. The order of the antipode of a finite dimensional Hopf algebra is finite. Amer. J. Math. 98 (1976), no. 2, 333–355.

[**Rad 1985**] Radford, David E. The structure of Hopf algebras with a projection. J. Algebra 92 (1985), no. 2, 322–347.

[**Rom-Rota 1978**] Roman, Steven M.; Rota, Gian-Carlo. The umbral calculus. Advances in Math. 27 (1978), no. 2, 95–188.

[**Rosso 1995**] Rosso, Marc. Quantum groups and braid groups. Symtries quantiques (Les Houches, 1995), 757–785, North-Holland, Amsterdam, 1998.

[**Rosso 1998**] Rosso, Marc. Quantum groups and quantum shuffles. Invent. Math. 133 (1998), no. 2, 399–416.

[**Rota 1978**] Rota, Gian-Carlo. Hopf algebra methods in combinatorics. Problmes combinatoires et thorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), pp. 363–365, Colloq. Internat. CNRS, 260, CNRS, Paris, 1978.

[**Sul 1971**] Sullivan, John Brendan. The uniqueness of integrals for Hopf algebras and some existence theorems of integrals for commutative Hopf algebras. J. Algebra 19 1971 426–440.

[**Sul 1973**] Sullivan, John Brendan. A decomposition theorem for pro-affine solvable algebraic groups over algebraically closed fields. Amer. J. Math. 95 (1973), 221–228.

[**Swe 1967**] Sweedler, Moss E. Cocommutative Hopf algebras with antipode. Bull. Amer. Math. Soc. 73 1967 126–128.

[**Swe 1969a**] Sweedler, Moss Eisenberg. Connected fully reducible affine group schemes in positive characteristic are abelian. J. Math. Kyoto Univ. 11 1971 51–70.

[**Swe 1969b**] Sweedler, Moss Eisenberg. Integrals for Hopf algebras. Ann. of Math. (2) 89 1969 323–335.

[**Swe 1969c**] Sweedler, Moss E. Hopf algebras. Mathematics Lecture Note Series W. A. Benjamin, Inc., New York 1969 vii+336 pp.

[**Taft 1971**] Taft, Earl J. The order of the antipode of finite-dimensional Hopf algebra. Proc. Nat. Acad. Sci. U.S.A. 68 1971 2631–2633.

[**Tak 1971**] Takeuchi, Mitsuhiro. Free Hopf algebras generated by coalgebras. J. Math. Soc. Japan 23 (1971), 561–582.

[**Tak 1972a**] Takeuchi, Mitsuhiro. A correspondence between Hopf ideals and sub-Hopf algebras. Manuscripta Math. 7 (1972), 251–270.

[**Tak 1972b**] Takeuchi, Mitsuhiro. On a semi-direct product decomposition of affine groups over a field of characteristic 0. Thoku Math. J. (2) 24 (1972), 453–456.

[**Yetter 1990**] Yetter, David N. Quantum groups and representations of monoidal categories. Math. Proc. Cambridge Philos. Soc. 108 (1990), no. 2, 261–290.