MATH 215 # Final Examination **Radford 08/08/08**

Name (print) ________________________________

(1) With the exception of part (a) of Problem 1, write your answers in the exam booklet provided. (2) *Return* this exam copy with your test booklet. (3) *You are expected to abide by the University's rules concerning academic honesty.*

---

1. (**25 points**) Let P and Q be statements.

a) (**8**) Complete the following three truth tables (on this sheet if you wish):

P	Q	P implies (P implies Q)
T	T	
T	F	
F	T	
F	F	

P	Q	P and (not Q)
T	T	
T	F	
F	T	
F	F	

P	Q	(not P) or Q
T	T	
T	F	
F	T	
F	F	

(b) (**11**) What are the logical relationships (implication, equivalence, negation) between the statements "P implies (P implies Q)", "P and (not Q)", and "(not P) or Q"? For convenience you may refer to these statements as A, B, and C respectively. Justify your answer in terms of the truth tables of parts (a).

(c) (**6**) What is the negation of "P and Q" in terms of "not P", "not Q"? What is the negation of "P or Q" in terms of "not P", "not Q"?

2. (**30 points**) Consider the statement "$a^2 - 5a - 14 > 0$ implies $a \leq -2$ or $7 \leq a$".

(a) (**5**) What is the *converse* of the statement?

(b) (**5**) What is the *contrapositive* of the statement? Write it without "not".

(c) (**10**) Prove the statement by contradiction. Clearly state your assumptions.

(d) (**10**) Prove the *contrapositive* of the statement directly. Clearly state your assumptions.

Base the proofs for parts (c) and (d) of Problem 2 on the axioms for the real number system $\mathbf{R}$ and for all for $a, b, c \in \mathbf{R}$: if $a, b > 0$ or $a, b < 0$ then $ab > 0$; if $a > 0$ and $b < 0$, or $a < 0$ and $b > 0$, then $ab < 0$; $a0 = 0 = 0a$; and if $a < b$ then $a - c < b - c$ and $c - a > c - b$.

3. (**15 points**) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a function and $a, L \in \mathbf{R}$. Then "$\lim_{x \longrightarrow a} f(x) = L$" can be expressed in terms of quantifiers by "$\forall \epsilon > 0, \exists \delta > 0, \forall x \in \mathbf{R}, 0 < |x-a| < \delta$ implies $|f(x)-L| < \epsilon$". (It is understood that $\epsilon, \delta \in \mathbf{R}$.)

(a) (**5**) Express the statement "$\lim_{x \longrightarrow a} f(x) = L$" in English with the quantifiers translated.

(b) (**10**) Express the statement "$\lim_{x \longrightarrow a} f(x) \neq L$", that is "not ($\lim_{x \longrightarrow a} f(x) = L$)", in terms of quantifiers without the use of "not".

4. (**25 points**) Show, by induction, that the sum of the squares of first n odd integers is given by the formula $1^2+3^2+5^2+\cdots+(2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$ for all $n \geq 2$ ($n=2$ is the base case).

5. (**30 points**) Let X, Y, and Z be sets.

(a) (**8**) Give the conditional definitions of $X-Y$ and $X\times Y$.

(b) (**5**) In terms of X and Y, what does it mean for $x \notin X-Y$?

(c) (**10**) Show that $X-(Y-Z) \subseteq (X-Y)\cup Z$.

(d) (**7**) Find subsets X, Y, and Z of $\{1,2,3\}$ such that $(X-Y)\cup Z \subseteq X-(Y-Z)$ is false. Compute the two preceding sets for your example.

6. (**25 points**) Suppose that $f : X \longrightarrow Y$ and $g : Y :\longrightarrow Z$ are functions.

(a) (**5**) Define, using quantifiers, what it means for f to be an *injection.*

(b) (**5**) Define, using quantifiers, what it means for f to be a *surjection.*

(c) (**5**) Suppose that $g\circ f$ is a surjection. Show that g is a surjection.

(d) (**10**) Let $A, B \subseteq Y$. Show that $\overleftarrow{f}(A\cup B) = \overleftarrow{f}(A)\cup\overleftarrow{f}(B)$. [For all $C \subseteq Y$ recall that $\overleftarrow{f}(C) = \{x \in X \mid f(x) \in C\}$.]

7. (**30 points**) *In this problem all binomial symbols must be computed.* A committee of 11 is to be formed from a group of 14 people.

(a) (**5**) How many such committees are there?

(b) (**5**) Suppose a certain 2 individuals from this group are to be *excluded.* How many such committees are there?

(c) (**5**) Suppose that a certain 5 individuals from this group are to be *included.* How many such committees are there?

Suppose A and B are different individuals in this group. Let X be the set of committees which include A and let Y the set of committees which include B.

(d) (**7**) Use the Principle of Inclusion-Exclusion to calculate the number of elements of $X\cup Y$, the set of committees which *include* A or B.

(e) (**8**) Let U be the set of all committees of 11 which can be formed from the group of 14 people. Express the set of committees which *exclude* both A and B in terms of X and Y and use De Morgan's Law to compute the number of such committees.

8. (**20 points**) Give the definition of (a) (**3**) finite set, (b) (**3**) denumerable set, (c) (**3**) countable set, and (d) (**11**) use the Euclidean Algorithm to find the greatest common divisor of 372 and 58.