Math 215 **Written Homework 7 Solution** **08/04/08**

1. (**20 points total**) The number of m-element subset of an n-element set, where $0 \le m \le n$, is $\binom{n}{m} = \frac{n!}{m!(n-m)!}$.

(a) $\binom{10}{6} = \frac{10!}{6!4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 3 \cdot 7 = 210$. (**3 points**)

(b) Since two particular individuals are to be included on the committee, these committees are formed by choosing $6 - 2 = 4$ from the remaining $10 - 2 = 8$. Thus the number is $\binom{10-2}{6-2} = \binom{8}{4} = \frac{8!}{4!4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 2 \cdot 5 = 70$. (**3 points**)

(c) Since two particular individuals are to be excluded from the committee, these committees are formed by choosing 6 from the remaining $10 - 2 = 8$. Thus the number is $\binom{10-2}{6} = \binom{8}{6} = \frac{8!}{6!2!} = \frac{8 \cdot 7}{2 \cdot 1} = 4 \cdot 7 = 28$. (**3 points**)

(d) See part (c). Thus the number is $\binom{10-1}{6} = \binom{9}{6} = \frac{9!}{6!3!} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 3 \cdot 4 \cdot 7 = 84$. (**3 points**)

(e) Let X be the set of committees of 6 with the first individual excluded and Y be the set of committees of 6 with the second excluded. Then $X \cup Y$ is the set of committees with one or the other excluded and $X \cap Y$ is the set of committees with both excluded. Thus

$|X \cup Y| = |X| + |Y| - |X \cap Y|$ (**4 points**) $= 84 + 84 - 28 = 140$. (**4 points**)

2. (**20 points total**) This exercise is best done by systematic listings.

(a) Isomorphisms $f : \{3, 5, 7\} \longrightarrow \{3, 5, 7\}$:

x	3	5	7
$f_1(x)$	3	5	7

x	3	5	7
$f_2(x)$	3	7	5

x	3	5	7
$f_3(x)$	5	3	7

x	3	5	7
$f_4(x)$	5	7	3

x	3	5	7
$f_5(x)$	7	3	5

x	3	5	7
$f_6(x)$	7	5	3

Note that each of these functions is its own inverse, except for f_4 and f_5 which are inverses of each other. (**8 points**)

(b) Surjections $f : \{3, 5, 7\} \longrightarrow \{a, b\}$:

x	3	5	7
$f_1(x)$	a	b	b

x	3	5	7
$f_2(x)$	b	a	b

x	3	5	7
$f_3(x)$	b	b	a

x	3	5	7
$f_4(x)$	b	a	a

x	3	5	7
$f_5(x)$	a	b	a

x	3	5	7
$f_6(x)$	a	a	b

(**6 points**)

(c) Injections $f : \{a, b\} \longrightarrow \{3, 5, 7\}$:

x	a	b
$f_1(x)$	3	5

x	a	b
$f_2(x)$	3	7

x	a	b
$f_3(x)$	5	3

x	a	b
$f_4(x)$	5	7

x	a	b
$f_5(x)$	7	3

x	a	b
$f_6(x)$	7	5

(**6 points**)

3. (**20 points total**) Let X be the set of residents of this small town.

(a) For $x \in X$ let $f(x)$ be the number of denominations resident x is carrying. Then $f(x) \in \{0, 1, \ldots, 6\}$ as there are 6 denominations. The question can be rephrased as how large does $|X|$ have to be to guarantee that $f(x_1) = f(x_2)$ for some $x_1 \neq x_2$; that is for f not to be injective. Answer: $|X| > 7$. (**10 points**)

(b) Let $f(x)$ be the set of types of denominations resident x is carrying. Then $f(x) \in P(\{\$1, \$5, \$10, \$20, \$50, \$100\})$ which has $2^6 = 64$ elements. In light of the solution to part (a), $|X| > 64$. (**10 points**)

4. (**20 points total**) f is surjective. Suppose $n \in \mathbf{Z}^+$. If $n = 2m$ for some $m \in \mathbf{Z}$, then $m > 0$ and thus $f(m) = 2m = n$ by definition of f. If $n = 2m + 1$ for some $m \in \mathbf{Z}$, then $m \geq 0$ which means $-m \leq 0$ and $f(-m) = -2(-m) + 1 = 2m + 1 = n$ by definition of f. We have shown that f is surjective. (**8 points**)

f is injective. Let $n, n' \in \mathbf{Z}$ and suppose that $f(n) = f(n')$.

Case 1: $f(n)$ is even. Then so is $f(n')$ and thus $n, n' > 0$ and $2n = f(n) = f(n') = 2n'$. But $2n = 2n'$ implies $n = n'$. (**6 points**)

Case 2: $f(n)$ is not even. Consequently $f(n)$ is odd. Then so is $f(n')$ and thus $n, n' \leq 0$ and $-2n + 1 = f(n) = f(n') = -2n' + 1$. But then $2(-n) + 1 = 2(-n') + 1$ which implies $n = n'$. (**6 points**)

We have shown in all cases that $f(n) = f(n')$ implies $n = n'$. Therefore f is injective.

5. (**20 points total**) The Principle of Inclusion-Exclusion: If X, Y are finite sets then $|X \cup Y| = |X| + |Y| - |X \cap Y|$.

(a) Using DeMorgan's Law $|A^c \cap B^c| = |(A \cup B)^c| = |U| - |A \cup B|$. (**4 points**) Since $|A \cup B| = |A| + |B| - |A \cap B| = 8 + 7 - 3 = 12$ (**4 points**) and $|U| = 21$, $|A^c \cap B^c| = 21 - 12 = 9$. (**2 points**)

(b) Let S and C be the sets of square tiles and circular tiles respectively, and let Let R and G be the sets of red tiles and green tiles respectively. Let U be the set of tiles. Then $S \cup C = U = G \cup R$ and these are disjoint unions. Thus by the Addition Principle (a special case of the Inclusion-Exclusion Principle)

$$|S| + |C| = |U| = |G| + |R|.$$

Since we are given that $|U| = 22$, $|S| = 9$, and $|R| = 11$, we conclude that $|C| = 13$ and $|G| = 11$.

(i) $|S \cup G| = |S| + |G| - |S \cap G| = 9 + 11 - 6 = 14$ (**3 points**) as $|S \cap G| = 6$ (given).

(ii) $S = (S \cap G) \cup (S \cap R)$ and is a disjoint union. Therefore

$$|S| = |S \cap G| + |S \cap R|,$$

or $9 = 6 + |S \cap R|$ which means $|S \cap R| = 3$.

Now $R = (R \cap S) \cup (R \cap C)$ and is a disjoint union. Therefore

$$|R| = |S \cap R| + |C \cap R|,$$

or $11 = 3 + |C \cap R|$ which means $|C \cap R| = 8$. (**3 points**)

(iii) $|C \cup R| = |C| + |R| - |C \cap R| = 13 + 11 - 8 = 16$. (**4 points**)