1. (25 points total)
   (a) Suppose \( n \) is a perfect square. Then \( n = a^2 \) for some \( a \in \mathbb{Z} \). Write \( a = 5\ell + r \), where \( \ell, r \in \mathbb{Z} \) and \( 0 \leq r < 5 \). (3 points) Now
   
   \[
   n = a^2 = (5\ell + r)^2 = 5^2\ell^2 + 2 \cdot 5\ell r + r^2 = 5(5\ell^2 + 2\ell r) + r^2. 
   \]
   (3 points)

   If \( r = 0, 1, \) or \( 2 \) then \( n = 5m, 5m + 1, \) or \( 5m + 4 \) respectively with \( m = 5\ell^2 + 2\ell r \). (3 points) If \( r = 3 \) then \( r^2 = 9 = 5 \cdot 1 + 4 \) so \( n = 5m + 4 \), where \( m = 5\ell^2 + 2\ell r + 1 \). (3 points) If \( r = 4 \) then \( r^2 = 16 = 5 \cdot 3 + 1 \) so \( n = 5m + 1 \), where \( m = 5\ell^2 + 2\ell r + 3 \). (3 points)

   (b) 288 = 5 \cdot 57 + 3; thus 288 is not a perfect square by part (a). (5 points)

   (c) 2369 = 5 \cdot 473 + 4; thus the test for perfect square by part (a) is inconclusive. However 2369 = 3 \cdot 789 + 2; thus 2369 is not a perfect square by Proposition 15.2.3. (5 points)

2. (25 points total) Suppose that \( n \in \mathbb{Z} \) and 7 divides \( n^2 \). Write \( n = 7m + r \), where \( m, r \in \mathbb{Z} \) and \( 0 \leq r < 7 \). Then \( n^2 = (7m + r)^2 = 7^2m^2 + 2 \cdot 7mr + r^2 = 7(7m^2 + 2mr) + r^2 \). Since 7 divides \( n^2 \) necessarily 7 divides \( r^2 \). (15 points)

   Using the table

   \[
   \begin{array}{c|ccccccc}
   r & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
   \hline
   r^2 & 0 & 1 & 4 & 9 & 16 & 25 & 36 \\
   \end{array}
   \]

   we see that 7 divides \( r^2 \) only when \( r = 0 \). Therefore \( r = 0 \) and \( n = 7m \). (10 points)

3. (25 points total) (a) 293 = 27 \cdot 10 + 23; \( q = 10 \) and \( r = 23 \) (10 points)
   (b) \(-2931 = 17 \cdot (-173) + 10; q = -173 \) and \( r = 10 \). (15 points)

4. (25 points total)
   (a)
\[\begin{align*}
89 &= 17 \cdot 5 + 4 \\
17 &= 4 \cdot 4 + 1 \\
4 &= 1 \cdot 4 + 0
\end{align*}\]

Therefore the greatest common divisor of 89 and 17 is 1. (15 points)

(b)

\[\begin{align*}
298 &= 8 \cdot 37 + 2 \\
8 &= 2 \cdot 4 + 0
\end{align*}\]

Therefore the greatest common divisor of 298 and 8 is 2. (10 points)