MATH 215 **Hour Exam II Solution** **Radford 08/01/08**

Name (print) ______________________________

(1) There are *four questions* on this exam. (2) *Return* this exam copy with your test booklet. (3) *You are expected to abide by the University's rules concerning academic honesty.*

Comment: Problems 1 and 3 should have been anticipated and were meant to be solved very quickly. Problem 3 was about basic definitions and simple proofs. Problem 4 was meant to be a bit of a challenge with its negations.

Some advice for the final examination: Know basic definitions and how to work with them, know how to construct simple proofs.

1. (20 pts.) For sets $A_1, \ldots, A_n$ define $A_1 \cup \cdots \cup A_n$ and $A_1 \cap \cdots \cap A_n$ inductively by

$$A_1 \cup \cdots \cup A_n = \begin{cases} A_1 & : \; n = 1; \\ (A_1 \cup \cdots \cup A_{n-1}) \cup A_n & : \; n > 1. \end{cases}$$

and

$$A_1 \cap \cdots \cap A_n = \begin{cases} A_1 & : \; n = 1; \\ (A_1 \cap \cdots \cap A_{n-1}) \cap A_n & : \; n > 1. \end{cases}$$

respectively. Suppose U is a universal set and $A_1, \ldots, A_n \subseteq U$. Use De Morgan's Laws and the above definitions to construct a proof by induction that $(A_1 \cap \cdots \cap A_n)^c = A_1^c \cup \cdots \cup A_n^c$. You may assume $A_1 \cup \cdots \cup A_n$, $A_1 \cap \cdots \cap A_n \subseteq U$. *Steps of your proof must be at least implicitly justified.*

Solution: Let $n \geq 1$ and $A_1, \ldots, A_n \subseteq U$. We show by induction that $(A_1 \cap \cdots \cap A_n)^c = A_1^c \cup \cdots \cup A_n^c$.

Suppose that $n = 1$. Then $(A_1 \cap \cdots \cap A_n)^c = (A_1)^c$ and $A_1^c \cup \cdots \cup A_n^c = A_1^c$. Since these are the same sets, the assertion is true for $n = 1$. (**4 points**)

Suppose that $n \geq 1$ and the assertion is true for all $A_1, \ldots, A_n \subseteq U$. Let $A_1, \ldots, A_{n+1} \subseteq U$. Then

$$\begin{aligned} (A_1 \cap \cdots \cap A_{n+1})^c &= ((A_1 \cap \cdots \cap A_n) \cap A_{n+1})^c \\ &= (A_1 \cap \cdots \cap A_n)^c \cup A_{n+1}^c \qquad \text{(by De Morgan's Law)} \\ &= (A_1^c \cup \cdots \cup A_n^c) \cup A_{n+1}^c \qquad \text{(by the induction hypothesis)} \\ &= A_1^c \cup \cdots \cup A_{n+1}^c. \end{aligned}$$

(**16 points**)

Comment The explicit justifications were not necessary, but all of the equations were.

2. (25 pts.) Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are functions.

(a) Suppose that f, g are surjections. Show that $g \circ f : X \longrightarrow Z$ is a surjection.

Solution: Let $z \in Z$. We need to show that there is an $x \in X$ such that $(g \circ f)(x) = z$.

Since g is a surjection, there is a $y \in Y$ such that $z = g(y)$. Since f is a surjection, there is an $x \in X$ such that $y = f(x)$. Since $(g \circ f)(x) = g(f(x)) = g(y) = z$ it follows that $g \circ f$ is a surjection. (**8 points**)

Comment: This proof involves nothing more than the definitions of surjection and composition of function. Here stating the assumptions and what is to be shown directs the proof.

Comment: In terms of quantifiers, f a surjection is described by "$\forall y \in Y, \exists x \in X, f(x) = y$." A few students reversed the quantifiers. "$\exists y \in Y, \forall x \in X, f(x) = y$" is to say that f is the constant function $f = y$. Such a function is surjective if and only if its codomain has one element.

(b) Let $A, B \subseteq Y$. Show that $\overleftarrow{f}(A \cap B) = \overleftarrow{f}(A) \cap \overleftarrow{f}(B)$. [Recall $\overleftarrow{f}(A) = \{x \in X \mid f(x) \in A\}$.]

Solution: **Note that** $\overleftarrow{f}(C) \subseteq X$ **for all** $C \subseteq Y$. We need to show that $x \in \overleftarrow{f}(A \cap B)$ implies $x \in \overleftarrow{f}(A) \cap \overleftarrow{f}(B)$ and vice versa.

Suppose that $x \in \overleftarrow{f}(A \cap B)$. Then $f(x) \in A \cap B$ by definition. Therefore $f(x) \in A$ and $f(x) \in B$. By definition $x \in \overleftarrow{f}(A)$ and $x \in \overleftarrow{f}(B)$. Thus $x \in \overleftarrow{f}(A) \cap \overleftarrow{f}(B)$. (**5 points**)

Conversely, suppose $x \in \overleftarrow{f}(A) \cap \overleftarrow{f}(B)$. Then $x \in \overleftarrow{f}(A)$ and $x \in \overleftarrow{f}(B)$. By definition $f(x) \in A$ and $f(x) \in B$. Therefore $f(x) \in A \cap B$ which means $x \in \overleftarrow{f}(A \cap B)$ by definition. (**5 points**)

Comment: This problem is basically an exercise in showing that two sets are equal. The sets involved are given by a conditional definition. The "by definition" above refers to the conditional definition.

(c) Suppose that $X = \{1, 2, 3, 4\}$, Y is the 4-element set $Y = \{a, b, c, d\}$, and f is given by the table

x	1	2	3	4
$f(x)$	c	b	a	d

. Describe the inverse f^{-1} by a similar table.

Solution: $f^{-1} : Y \longrightarrow X$ and $f^{-1}(y) = x$ if and only if $f(x) = y$. Thus:

y	a	b	c	d
$f^{-1}(y)$	3	2	1	4

.

(**7 points**) Also

y	c	b	a	d
$f^{-1}(y)$	1	2	3	4

.

3. (30 pts.) *In this problem binomial symbols must be computed.* A committee of 8 persons is to be formed a group of 11 people.

(a) Find the number of such committees.

Solution: $\binom{11}{8} = \dfrac{11{\cdot}10{\cdot}9}{3{\cdot}2{\cdot}1} = 11{\cdot}5{\cdot}3 = 165$. (**7 points**)

(b) Find the number of such committees, given that a particular individual is to be *included*.

Solution: $\binom{11-1}{8-1} = \binom{10}{7} = \frac{10{\cdot}9{\cdot}8}{3{\cdot}2{\cdot}1} = 10{\cdot}3{\cdot}4 = 120.$ (**6 points**)

(c) Find the number of such committees, given that a particular individual is to be *excluded*.

Solution: $\binom{11-1}{8} = \binom{10}{8} = \frac{10{\cdot}9}{2{\cdot}1} = 5{\cdot}9 = 45.$ (**7 points**)

(d) Use the Principle of Inclusion-Exclusion to find the number of such committees, given that at least one of two particular individuals is to be *excluded*.

Specific instructions for part (d): Let A and B be these individuals, let X be the set of committees which exclude A, and let Y be the set of committees which exclude B. Express the set of committees of part (d) in terms of X and Y and count them.

Solution: $|X \cup Y| = |X| + |Y| - |X \cap Y| = \binom{10}{8} + \binom{10}{8} - \binom{9}{8} = 45 + 45 - 9 = 81.$ (**10 points**)

Comment: There is another way of computing $|X \cup Y|$. This solution would *not* be acceptable for part (d).

Let U be the set of committees of the 11 consisting of 8 people. Then $|U| = \binom{11}{8}$. Note that $(X \cup Y)^c = X^c \cap Y^c$ is the set of committees of 11 consisting of 8 people which *include* both A and B. Since $X \cup Y = U - (X \cup Y)^c$,

$$|X \cup Y| = |U| - |(X \cup Y)^c| = \binom{11}{8} - \binom{11-2}{8-2} = \binom{11}{8} - \binom{9}{6} = 165 - 84 = 81$$

4. (25 pts.) Let A, B, and C be sets.

(a) The conditional definition of the intersection of A and B is $A \cap B = \{x \,|\, x \in A \text{ and } x \in B\}$. Give conditional definitions of $A \cup B$ and $A - B$.

Solution: $A \cup B = \{x \,|\, x \in A \text{ or } x \in B\}$. (**3 points**) $A - B = \{x \,|\, x \in A \text{ and } x \notin B\}$. (**3 points**)

(b) Show that $A - (B - C) \subseteq (A - B) \cup C$. [Hint: $x \notin B - C$, that is "not $(x \in B - C)$", if and only if ?]

Solution: From part (a) we see that:

"$x \notin B - C$" is logically equivalent to "$(x \notin B)$ or $(x \in C)$".

Very, very important:

“not (P and Q)” and “(not P) or (not Q)” are logically equivalent;

“not (P or Q)” and “(not P) and (not Q)” are logically equivalent.

Variations of De Morgan’s Laws. To the solution of part (b).

Let $x \in A - (B - C)$. We need to show that $x \in (A - B) \cup C$. We have the following equivalences:

$$
\begin{aligned}
&x \in A - (B - C) \\
&(x \in A) \text{ and } (x \notin B - C) \\
&(x \in A) \text{ and } ((x \notin B) \text{ or } (x \in C)) \\
&((x \in A) \text{ and } (x \notin B)) \text{ or } ((x \in A) \text{ and } (x \in C));
\end{aligned}
$$

the latter follows since “and” distributes over “or” and implies “$(x \in A)$ and $(x \notin B))$ or $(x \in C)$”, or equivalently “$x \in (A - B) \cup C$”. (**10 points**)

(c) Show that $A - (B - C) = (A - B) \cup C$ if $C \subseteq A$.

Solution: In light of part (b) we need only show that $x \in (A - B) \cup C$ implies $x \in (A - (B - C)$. Suppose that $x \in (A - B) \cup C$. Then $x \in A - B$ or $x \in C$.

Case 1: $x \in A - B$. In this case $x \in A$ and $x \notin B$. Therefore $x \notin B - C$ by the boxed statement; thus $x \in A - (B - C)$.

Case 2: $x \in C$. Then $x \notin B - C$ by the boxed statement. Since $C \subseteq A$ by assumption, $x \in A$. Therefore $x \in A - (B - C)$.

In any event $x \in A - (B - C)$. (**9 points**)

Comment: There is an analogy between the equation $a - (b - c) = (a - b) + c$ for real numbers and the equation $A - (B - C) = (A - B) \cup C$ for sets, which holds when $C \subseteq A$.