1. **20 points total** We prove the formula \( \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \) holds for all \( n \geq 1 \) by induction on \( n \).

   Suppose that \( n = 1 \). Then \( \sum_{i=1}^{1} i^3 = 1^3 = 1 \) and \( \frac{n^2(n+1)^2}{4} = \frac{1^2(1+1)^2}{4} = \frac{1 \cdot 4}{4} = 1 \).

   Therefore the formula holds when \( n = 1 \). (5 points)

   Suppose that \( n \geq 1 \) and the assertion holds for \( n \); that is \( \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \). We will show that the assertion holds for \( n + 1 \); that is \( \sum_{i=1}^{n+1} i^3 = \frac{(n+1)^2((n+1)+1)^2}{4} \), or equivalently \( \sum_{i=1}^{n+1} i^3 = \frac{(n+1)^2(n+2)^2}{4} \).

   Since

   \[
   \sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + (n+1)^3 \quad (5 \text{ points})
   \]

   \[
   = \frac{n^2(n+1)^2}{4} + (n+1)^3 \quad (5 \text{ points})
   \]

   \[
   = \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \quad (5 \text{ points})
   \]

   \[
   = \frac{(n+1)^2[n^2 + 4(n+1)]}{4} \quad (5 \text{ points})
   \]

   \[
   = \frac{(n+1)^2(n^2 + 4n + 4)}{4} \quad (5 \text{ points})
   \]

   \[
   = \frac{(n+1)^2(n+1)^2}{4} \quad (5 \text{ points})
   \]

   We have shown that if the formula is true for \( n \geq 1 \) then it is true for \( n + 1 \). Therefore the formula is true for all \( n \geq 1 \).

2. **20 points total** Part (a) is to make part (b) easier.

   (a) The statement we are to prove is \( n + 1 \leq 2^{n-1} \) for \( n \geq 3 \). When \( n = 3 \), \( n + 1 = 4 = 2^2 = 2^{3-1} = 2^{n-1} \), thus the statement is true in the base case. (2 points)

   Suppose \( n \geq 3 \) and the statement is true. The statement for \( n+1 \) is \((n+1)+1 < 2^{(n+1)-1}\) or \((n+1) + 1 < 2^n\). The calculation

   \[
   (n + 1) + 1 < (n + 1) + (n + 1) \leq 2^{n-1} + 2^{n-1} = 2 \cdot 2^{n-1} = 2^{n} \quad (3 \text{ points})
   \]

   shows that the statement holds for \( n + 1 \). Therefore the statement is true for all \( n \geq 3 \). (3 points)
(b) The statement we are to prove is $n^2 < 2^n$ for $n > 4$, or equivalently $n \geq 5$. When $n = 5$, $n^2 = 5^2 = 25 < 32 = 2^5$; thus the statement holds for $n = 5$, the base case. (2 points)

Suppose that $n \geq 5$ and the statement is true. The statement for $n+1$ is $(n+1)^2 < 2^{n+1}$.

Using part (a) (2 points) we calculate

$$(n+1)^2 = n^2 + 2n + 1 < n^2 + 2(n+1) < 2^n + 2\cdot 2^{n-1} = 2\cdot 2^n = 2^{n+1}$$

which means that the statement is true for $n + 1$. Thus the statement holds for all $n \geq 5$. (2 points)

(c) $n = 1$: $1^2 < 2 = 2^1$, true; $n = 2$: $2^2 = 4 \not< 4 = 2^2$, false; $n = 3$: $3^2 = 9 \not< 8 = 2^3$, false; $n = 34$: $4^2 = 16 \not< 16 = 2^4$, false; $n \geq 5$, true by part (a). $n = 2, 3, 4$. (3 points)

3. **20 points total** “$A \cup B \subseteq A$ only if $B \subseteq A$” is the same as “$A \cup B \subseteq A$ implies $B \subseteq A$”.

Assume that $A \cup B \subseteq A$. The implication follows once we show $B \subseteq A$. Let $x \in B$. (5 points) Then $x \in A$ or $x \in B$ which means $x \in A \cup B$ by definition of union. (5 points) Since $A \cup B \subseteq A$, $x \in A$ by definition of set inclusion. We have shown that $x \in B$ implies $x \in A$. (5 points) Therefore $B \subseteq A$. (5 points)

4. **40 points total** “$B \subseteq A \cap B$ if and only if $B \subseteq A$”.

**Only if**: ‘$B \subseteq A \cap B$ implies $B \subseteq A$’. (5 points)

Assume $B \subseteq A \cap B$ and let $x \in B$. (5 points) Since $B \subseteq A \cap B$, $x \in A \cap B$. Thus $x \in A$ and $x \in B$; in particular $x \in A$. (5 points) We have shown $x \in B$ implies $x \in A$. Therefore $B \subseteq A$. (5 points)

**If**: “$B \subseteq A$” implies “$B \subseteq A \cap B$”. (5 points)

Assume that $B \subseteq A$ and let $x \in B$. (5 points) Then $x \in A$, since $x \in B$, which means $x \in A$ and $x \in B$. Therefore $x \in A \cap B$. (5 points) We have shown $x \in B$ implies $x \in A \cap B$. Therefore $B \subseteq A \cap B$. (5 points)