1. We define the Cartesian product of sets $A_1, \ldots, A_n$ inductively by

$$A_1 \times \cdots \times A_n = \begin{cases} A_1 & : n = 1; \\ (A_1 \times \cdots \times A_{n-1}) \times A_n & : n > 1. \end{cases}$$

Suppose that $A_1, \ldots, A_n$ are finite sets. Show, by induction, that $|A_1 \times \cdots \times A_n| = |A_1| \cdot \cdots \cdot |A_n|$ for $n \geq 1$. (You may assume this is the case for $n = 2$.)

2. What are the logical relationships between the following statements:
   (a) $\exists a \in A, \exists b \in B$, not $P(a, b)$;
   (b) $\forall a \in A, \forall b \in B$, $P(a, b)$;
   (c) $\forall a \in A, \exists b \in B$, $P(a, b)$;
   (d) $\exists a \in A, \forall b \in B$, not $P(a, b)$?

Comment: To show that one statement does not always imply another, supply a specific counterexample. Try $P(a, b) : a^2 = 5$ or $P(a, b) : b^3 = 7$ for example, where $A = B = \mathbb{R}$.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function and $a, b \in \mathbb{R}$. The very compact definition of $\lim_{x \to a} f(x) = b$ is

$$\forall \epsilon > 0, \exists \delta > 0, P(\epsilon, \delta),$$

where

$$P(\epsilon, \delta) : \forall x \in \mathbb{R}, (0 < |x - a| < \delta) \Rightarrow (|f(x) - b| < \epsilon).$$

3. Let $f(x) = \begin{cases} 3x + 5 & : x \neq 4 \\ -10 & : x = 4. \end{cases}$ Prove that $\lim_{x \to 4} f(x) = 17$.

4. We examine what it means for $\lim_{x \to a} f(x)$ not to exist.
(a) Using the compact definition of $\lim_{x \to a} f(x) = b$, express in the same manner “not (lim$_{x \to a} f(x) = b$)”.
Write “not $P(\epsilon, \delta)$” explicitly.

(b) Show that $\lim_{x \to 0} f(x) = b$ is false for all $b \in \mathbb{R}$, where $f(x) = \left\{ \begin{array}{ll} 1 & : x \geq 0 \\ -1 & : x < 0 \end{array} \right.$.

5. Let $f : X \to Y$ be a function. Then $f$ is surjective means $\forall y \in Y, \exists x \in X, f(x) = y$.

(a) In the style of the definition of surjective, express what it means for $f$ not to be surjective.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - 2x + 20$. Show that $f(x)$ is not surjective by showing that the condition of part (a) is satisfied.