Variations on the Induction Principle

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The purpose of these notes is to show that there are equivalent formulations of the Principle of Induction, Theorem 2 below. Consider the following statements:

Theorem 1 (Existence of a Least Element) *Every non-empty subset of the positive integers has a least element.*

Theorem 2 (Induction Principle) *For all positive integers n let $P(n)$ be a statement. Suppose that*

a) *$P(1)$ is true and*

b) *for all $n \geq 1$ if $P(n)$ is true then $P(n+1)$ is true.*

Then $P(n)$ is true for all $n \geq 1$.

Theorem 3 (Strong Induction Principle) *For all positive integers n let $P(n)$ be a statement. Suppose that*

a) *$P(1)$ is true and*

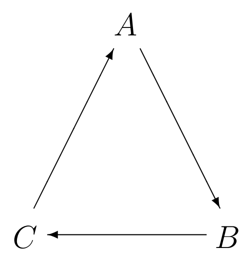
b) *for all $n \geq 1$ if $P(1), \ldots, P(n)$ are true then $P(n+1)$ is true.*

Then $P(n)$ is true for all $n \geq 1$.

We will show that the preceding theorems are logically equivalent. That is not to say that any of them is true, but any one of them is true then they all are true.

Theorem 4 *Theorems 1–3 are logically equivalent.*

Proof: We show that Theorem 1 implies Theorem 2, that Theorem 2 implies Theorem 3, and finally that Theorem 3 implies Theorem 1. Thus any one of Theorems 1–3, denoted by A–C respectively, implies any other since "implies" is transitive. This is easily seen from the diagram below, where "$\longrightarrow$" denotes "implies".



Suppose that Theorem 1 is true. We show that Theorem 2 is true.

Assume the hypothesis of Theorem 2 and let S be the subset of those positive integers n such that $P(n)$ is false. We need only show that $S = \emptyset$. For then $P(n)$ is true for all positive integers n; that is, Theorem 2 is true.

Suppose to the contrary that $S \neq \emptyset$. Then S has a least element n by Theorem 1. Since $P(1)$ is true by part a) of Theorem 2, $1 \notin S$. Therefore $n > 1$. Since $0 < n-1 < n$ it follows that $n-1$ is a positive integer and $n-1 \notin S$. By definition of S the statement $P(n-1)$ is true. By part b) of Theorem 2 we conclude that $P(n)$ is true. Thus $n \notin S$, a contradiction. Therefore $S = \emptyset$ and our proof that Theorem 1 implies Theorem 2 is complete.

Suppose that Theorem 2 is true. We show that Theorem 3 is true.

Assume the hypothesis of Theorem 3. Let $P'(1)$ be the statement $P(1)$ and for $n > 1$ let $P'(n)$ be the statement

$$P(1) \text{ and } P(2) \ \cdots \text{ and } P(n).$$

Observe that $P'(n)$ is true if and only if $P(1), P(2), \ldots, P(n)$ are true.

We need only show that the statements $P'(1), P'(2), \ldots$ satisfy the hypothesis of Theorem 2. For then $P'(n)$ is true for all $n \geq 1$. In particular $P(n)$ is true for all $n \geq 1$ and thus Theorem 3 is true.

Since $P(1)$ is true by part a) of Theorem 3 it follows that $P'(1)$ is true. Suppose that $n \geq 1$ and $P'(n)$ is true. Then $P(1), \ldots, P(n)$ are true. By part b) of Theorem 2 the statement $P(n+1)$ is true. Therefore $P'(n+1)$ is true. We have shown that the hypothesis of Theorem 2 is satisfied for $P'(1), P'(2), \ldots$ which completes our proof that Theorem 2 implies Theorem 3.

Suppose that Theorem 3 is true. We show that Theorem 1 is true. This we do by establishing the *contrapositive* of the conclusion of Theorem 1; namely that if S is a subset of positive integers with no minimal element then $S = \emptyset$.

Suppose that S is a subset of positive integers with no minimal element. For $n \geq 1$ let $P(n)$ be the statement $n \notin S$. Since S has no minimal element $1 \notin S$. Therefore $P(1)$ is true.

Suppose that $n \geq 1$ and $P(1), \ldots, P(n)$ are true. Then none of $1, \ldots, n$ are in S. If $n+1 \in S$ then $n+1$ would be a minimal element of S, a contradiction. Therefore $n+1 \notin S$; that is $P(n+1)$ is true.

We have shown that $P(1), P(2), \ldots$ satisfy the hypothesis of Theorem 3. Therefore $P(n)$ is true for all $n \geq 1$; that is $n \notin S$ for all $n \geq 1$. In other words $S = \emptyset$ as required. Our proof that Theorem 3 implies Theorem 1 is complete. □

Exercises

These exercises are optional, of course.

Exercise 1 Consider the following (change of base for induction):

Theorem 5 *Let N be any integer and suppose that $P(n)$ is a statement for all $n \geq N$. Assume that*

a) *$P(N)$ is true and*

b) *for all $n \geq N$ if $P(n)$ is true then $P(n+1)$ is true.*

Then $P(n)$ is true for all $n \geq N$.

Show that Theorem 5 is logically equivalent to Theorems 1–3. [Hint: We need only show that Theorem 5 is equivalent to Theorem 2. Theorem 2 is the special case of Theorem 5 where $N = 1$. To show that Theorem 2 implies Theorem 5 let $P'(n)$ be the statement $P(N+n-1)$ for all $n \geq 1$. Thus $P'(1)$ is $P(N)$, $P'(2)$ is $P(N+1)$, $P'(3)$ is $P(N+2), \ldots$.]

Exercise 2 Show that Theorem 1 is logically equivalent to:

Theorem 6 *Suppose that N is a fixed integer. Then every non-empty subset of $\{n \in \mathbf{Z} \,|\, n \geq N\}$ has a least element.*