Math 215, Fall 05 Homework #2 Solution

09/14/05

There are usually several valid proofs for a given proposition. Your proofs do not necessarily have to match the ones below to be correct.

1. (**20 points total**) a) Let $a \in \mathbf{R}$. By Axiom 6 there *exists* an element $a' \in \mathbf{R}$ such that $a + a' = 0 = a' + a$. Suppose that $a'' \in \mathbf{R}$ also satisfies $a + a'' = 0 = a'' + a$. Then by Axioms 4 and 2 we can make the calculation

$$a'' = a'' + 0 = a'' + (a + a') = (a'' + a) + a' = 0 + a' = a'$$

which proves $a'' = a'$. Therefore there is *at most one* element in $\mathbf{R}$ which plays the role of a'. Existence plus uniqueness (at most one) is the same as "there exists exactly one", or "there exists a unique". (**Problem technically stated 2 points, correct algebra 4 points**.)

b) Let $a, b, x \in \mathbf{R}$ and suppose that $a + x = b$. Then

$$x = 0 + x = ((-a) + a) + x = (-a) + (a + x) = (-a) + b$$

which means that there is *at most one solution to* $a + x = b$ which must be $x = (-a) + b$. We have shown uniqueness. (**4 points.**) As for *existence*, note

$$a + ((-a) + b) = (a + (-a)) + b = 0 + b = b$$

which shows that $x = (-a) + b$ is indeed a solution to $a + x = b$. (**4 points**.)

c) Let $a \in \mathbf{R}$. Then $a0 = a(0 + 0) = a0 + a0$. Therefore $x = a0$ is a solution to $a0 + x = a0$, as is $x = 0$. But there is a unique solution to $a0 + x = a0$ by part b). Therefore $a0 = 0$. (**4 points**.) Since $0a = a0$ in any case, $0a = 0$ by our previous calculation. (**2 points**.)

Comment: Note that we did not make our reasons *explicit*. We wrote the proof so that each step reflected a single assumption or axiom. This is permissible. However "Adding a to both sides of $b = c$ gives $a + b = c + a$." is not permissible since there are two steps involved here; $a + b = a + c$ and $a + c = c + a$.

2. (**20 points total**) Let $a, b \in \mathbf{R}$. By part c) of Proposition 1 and the distributive law(s) we calculate

$$0 = a0 = a(b + (-b)) = ab + a(-b).$$

As $0 = ab+(-(ab))$ also, both $a(-b)$ and $-(ab)$ are solutions to the equation $ab + x = 0$. There is exactly one solution the this equation by part b) of Proposition 1; therefore $a(-b) = -(ab)$. (**Correct start, 5 points; algebra 10 points**.)

The proof that $(-a)b = -(ab)$ is similar. Better yet, using commutativity and what we have just proved we *deduce*

$$(-a)b = b(-a) = -(ba) = -(ab).$$

(**5 points**)

3. (**20 points total**) Suppose that $e, e' \in \mathbf{R}$ satisfy $ea = a = ae$ and $e'a = a = ae'$ for all $a \in \mathbf{R}$. With $a = e'$ the first pair of equations becomes $ee' = e' = e'e$ and with $a = e$ the second pair reads $e'e = e = ee'$. In particular $ee' = e'$ and $e = ee'$. Therefore $e' = e$.

Comment: Observe that *no* axioms are used in the proof, just the defining property of e and e'.

4. (**20 points total**) We first decide whether P implies Q is true and Q implies P is true. Note that Q can be reformulated $(a - 3)(a + 1) > 0$.

Suppose P is true; that is $a > 3$. Then $a - 3, a + 1 > 0$. Therefore the product $(a-3)(a+1) > 0$, or Q is true. Therefore P implies Q. On the other hand, if $a = -2$ then $a^2 - 2a - 3 = 5 > 0$. Therefore Q does not imply P.

a) P implies Q. *True*. (**6 points for justification**.)

b) Q only if P, or equivalently Q implies P. *False* (**5 points for justification**.)

c) P if and only if Q. *False* in light of part b). (**3 points**.)

d) P is necessary for Q, or equivalently Q implies P. *False* in light of part b). (**3 points**.)

e) P is sufficient for Q, or equivalently P implies Q. *True* in light of part a). (**3 points**.)