Math 215, Fall 05 Homework #3 Solution

09/21/05

1. (**20 points total**) Our assertion is: $0 < n < 3$ implies $n^3 < -n^2 + 12n$.

a) There are two cases.

Case 1: $n = 1$ in which case $1^3 = 1 < 11 = -1^2 + (12)(1)$.

Case 2: $n = 2$ in which case $2^3 = 8 < 20 = -2^2 + (12)(2)$. (**4 points**)

b) We construct a table for convenience.

$n^3 < -n^2 + 12n$
  is implied by
$0 < n$ and $n^2 < -n + 12$
  is implied by
$0 < n$ and $n^2 + n - 12 < 0$
  is implied by
$0 < n$ and $(n - 3)(n + 4) < 0$
  is implied by
$0 < n < 3$;

the last implication holds since $0 < n < 3$ implies $n - 3$ is negative and $n + 4$ is positive which means the product is negative. (**8 points**)

c) Under the assumption that $0 < n$, observe that the non-indented lines of the table are logical equivalences. Consider:

$n^3 < -n^2 + 12n$
  is implied by
$n < 0$ and $n^2 > -n + 12$
  is implied by
$n < 0$ and $n^2 + n - 12 > 0$
  is implied by
$n < 0$ and $(n - 3)(n + 4) > 0$
  is implied by
$n < -4$;

the last implication holds since $n < -4$ implies $n-3, n+4$ are negative which means the product is positive. (**8 points**)

Observe that under the assumption that $n < 0$, observe that the non-indented lines of the preceding table are logical equivalences. Now $n^3 < -n^2 + 12n$ implies $n \neq 0$. Thus $0 < n$ or $n < 0$. Therefore $n^3 < -n^2 + 12n$ is logically equivalent to $n < -4$ or $0 < n < 3$.

Comment: Part c) was misstated for which I apologize. As compensation I gave everyone 8 points for part c). (**8 points**)

2. (**20 points total**) Observe that the hypothesis $a^2 \geq 9a$ is equivalent to $a(a-9) \geq 0$. Thus we may rephrase the assertion: $a(a-9) \geq 0$ implies $a \leq 0$ or $9 \leq a$.

a) Suppose that the hypothesis $a(a-9) \geq 0$ is true and the conclusion $a \leq 0$ or $9 \leq a$ is false. Then $0 < a < 9$. Thus $a > 0$ and $a - 9 < 0$ which implies $a(a-9) < 0$, a contradiction of the hypothesis. Therefore the hypothesis implies the conclusion. (**10 points**)

Comment: Be very careful when substituting values in a proof by contradiction. Here is an example of a type of invalid proof which some students used.

Assertion: $a \geq 0$ implies $a > 0$.

"Proof:" Assume the hypothesis $a \geq 0$ is true and that the conclusion $a > 0$ is false. Then $a \leq 0$. With $a = -1$ we have a contradiction to $a \geq 0$. Therefore the assertion is proved by contradiction.

b) We consider two cases.

Case 1: $a = 0$ or $a = 9$. In either situation $a \leq 0$ or $9 \leq a$.

Case 2: Not Case 1; that is $a \neq 0$ and $a \neq 9$. Then $a, a - 0 \neq 0$. Since $a(a-9) \geq 0$, and neither factor is zero, $a(a-9) > 0$. Therefore both $a, a-9$ are positive or both are negative. In the first situation $9 < a$ and in the second $a < 0$. In any event $a \leq 0$ or $9 \leq a$. (**10 points**)

3. (**20 points total**) Observe that the hypothesis $a^2 - 12a + 32 < 0$ can be written $(a-4)(a-8) < 0$. We want to show that the hypothesis implies the conclusion.

a) Assume that the hypothesis is true and the conclusion is false. Then $(a-4)(a-8) < 0$ and either $a < 4$ or $8 < a$.

Case 1: $a < 4$. In this case $a-4, a-8 < 0$. Therefore $(a-4)(a-8) > 0$, a contradiction.

Case 2: $8 < a$. In this case $a-8, a-4 > 0$. Therefore $(a-4)(a-8) > 0$, a contradiction.

In any event the hypothesis is contradicted. Therefore the hypothesis implies the conclusion. (**8 points**)

b) Suppose that $a^2 - 12a + 32 < 0$, or equivalently $(a-4)(a-8) < 0$. Since the product of any number and zero is zero, $a \neq 4, 8$. Since the product of two non-zero numbers with same sign is positive, $a-4$ and $a-8$ must have different signs. This leads to two cases.

Case 1: $a-4 > 0$ and $a-8 < 0$, or equivalently $4 < a < 8$.

Case 2: $a-4 < 0$ and $a-8 > 0$, or equivalently $8 < a < 4$. This case is vacuous – that is nothing satisfies the condition.

Combining the results of the cases we have $4 < a < 8$. Therefore $4 \leq a \leq 8$. (**8 points**)

c) The converse of the assertion is $4 \leq a \leq 8$ implies $a^2 - 12a + 32 < 0$. Since $a^2 - 12a + 32 = (a-4)(a-8) = 0$ when $a = 4$ the converse is *false*. (**4 points**)

4. (**20 points total**) This problem is an exercise in definitions.

a) Let $a, b, c \in S$. Then

$$(a \cdot^{op} b) \cdot^{op} c = (b \cdot a) \cdot^{op} c = c \cdot (b \cdot a)$$

and

$$a \cdot^{op} (b \cdot^{op} c) = a \cdot^{op} (c \cdot b) = (c \cdot b) \cdot a.$$

Therefore $(a \cdot^{op} b) \cdot^{op} c = a \cdot^{op} (b \cdot^{op} c)$ for all $a, b, c \in S$ if and only if $c \cdot (b \cdot a) = (c \cdot b) \cdot a$ for all $c, b, a \in S$. (**8 points**)

Comment: Here is an argument for part a) in a bit more detail. We start with the calculations given above.

Suppose that the original product is associative. Then

$$(a \cdot^{op} b) \cdot^{op} c = c \cdot (b \cdot a) = (c \cdot b) \cdot a = a \cdot^{op} (b \cdot^{op} c)$$

for all $a, b, c \in S$. Therefore the new product is associative. (What is wrong with $(a \cdot^{op} b) \cdot^{op} c = c \cdot (b \cdot a) = a \cdot^{op} (b \cdot^{op} c) = (c \cdot b) \cdot a$?)

Suppose that the new product is associative. Then

$$c{\cdot}(b{\cdot}a) = (a{\cdot}^{op}b){\cdot}^{op}c = a{\cdot}^{op}(b{\cdot}^{op}c) = (c{\cdot}b){\cdot}a$$

for all $c, b, a \in S$. Therefore the original product is associative.

b) Let $a, b \in S$. Then

$$a{\cdot}^{op}b = b{\cdot}a$$

and

$$b{\cdot}^{op}a = a{\cdot}b.$$

Therefore $a{\cdot}^{op}b = b{\cdot}^{op}a$ for all $a, b \in S$ if and only if $b{\cdot}a = a{\cdot}b$ for all $b, a \in S$. (**6 points**)

c) Since $a{\cdot}e = a$ for all $a \in S$, with $a = e'$ the equation specializes to $e'{\cdot}e = e'$. Since $e'{\cdot}a = a$ for all $a \in S$, with $a = e$ the last equation specializes to $e'{\cdot}e = e$. Therefore $e' = e'{\cdot}e = e$. (**6 points**)

Comment: You can not assume the existence of inverses or associativity since they are not given as part of the assumptions.