

Math 215, Fall 05

Homework #4

Due Friday, 09/23/05 at the beginning of class

You may assume the propositions and theorems of the text. You may assume that the product of two positive real numbers, and the product of two negative real numbers, is positive. You may also assume that the product of a positive real number and a negative real number is negative.

1. Let a be a real number. Consider the assertion: $a^2 - 6a + 5 \geq 0$ implies $a \leq 1$ or $a \geq 5$.

- a) Prove the assertion by contradiction.
- b) Prove the assertion by assuming the hypothesis and showing that $a \not\leq 1$ implies $a \geq 5$.
- c) State the *contrapositive* of the assertion and prove the contrapositive directly. (Since the contrapositive is equivalent to the assertion, this gives another proof of the assertion.)
- d) Prove the *converse* of the assertion by cases.

2. We revisit part d) of the preceding problem. Let a be an *integer*. Prove that $a \geq 5$ implies $a^2 - 6a + 5 \geq 0$ by induction on a . [Hint: Base a proof on the calculation $(a + 1)^2 - 6(a + 1) + 5 = a^2 - 6a + 5 + d$, determine d , and show that $d \geq 0$ under appropriate assumptions.]

3. Prove the following by induction: $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all positive integers n . (When $n = 1$ the sum is taken to be 1^2 by convention.)

4. For a real number a and non-negative integer n we define the power a^n by

$$a^n = \begin{cases} 1 & : n = 0; \\ a^{n-1}a & : n \geq 1. \end{cases}$$

Let a be a real number and m be a non-negative integer.

- a) Show that $a^{m+n} = a^m a^n$ for all non-negative integers n by induction on n .
- b) Show that $(a^m)^n = a^{mn}$ for all non-negative integers n by induction on n .