1. (20 points total) The statements of parts b) and c) are logically equivalent. (12 points) The statement of part a) implies the statement of part b), and thus the statement of part c) also (8 points).

2. (20 points total)

a) Suppose that \( a, b > 0 \) and \( a^2 < b^2 \). Then \( a + b > 0 \) and \( 0 < b^2 - a^2 = (b-a)(b+a) \). If \( b-a = 0 \) then \( (b-a)(b+a) = 0 \), a contradiction. If \( b-a < 0 \) then \( (b-a)(b+a) < 0 \), a contradiction. Therefore \( b - a > 0 \) or equivalently \( b > a \). (6 points)

b) In any event \(|a| \geq 0\). If \( a \geq 0 \) then \(|a|=a\) by definition. If \( a < 0 \) then \( a < 0 \leq |a| \). In both cases \( a \leq |a| \). (2 points)

If \( a > 0 \) then \(|a|=a\) which means \(|a|^2 = a^2 \). If \( a < 0 \) then \(|a| = -a \) which means \(|a|^2 = (-a)^2 = (-a)(-a) = -(-aa) = a^2 \). (2 points)

c) In order to use the hint, one needs to show (or at least point out the use of) Equations (1) and (2) below.

\[
a^2 = b^2 \quad \text{implies} \quad a = b \quad \text{for all} \quad a, b \geq 0.
\] (1)

To prove this, assume that \( a, b \in \mathbb{R} \) are non-negative and \( a^2 = b^2 \). Then \( 0 = a^2 - b^2 = (a-b)(a+b) \) which means \( a - b = 0 \) or \( a + b = 0 \). In the first case \( a = b \). In the second \( a + b = 0 \). Here \( 0 \leq a = -b \leq 0 \) from which we conclude that \( 0 = a = -b \). Thus \( a = 0 = b \) in the second case. In any event \( a = b \).

As a consequence of (1) we have:

\[
|ab| = |a||b| \quad \text{for all} \quad a, b \in \mathbb{R}.
\] (2)
To see this we first note that $|ab|, |a|, |b| \geq 0$, and thus $|a||b| \geq 0$. By part b) we have $|ab|^2 = (ab)^2 = a^2b^2 = |a|^2|b|^2 = (|a||b|)^2$ which means $|ab| = |a||b|$ by (1). (2 points for all of the above)

We can now calculate, using part b) and (2),

$$|a + b|^2 = (a + b)^2 = a^2 + 2ab + b^2 \leq |a|^2 + 2|ab| + |b|^2 = |a|^2 + 2|a||b| + |b|^2 = (|a| + |b|)^2.$$  

Therefore $|a+b|^2 \leq (|a|+|b|)^2$. Thus $|a+b|^2 \leq (|a|+|b|)^2$. As $|a+b|, |a|, |b| \geq 0$ and hence $|a| + |b| \geq 0$, using part a) and (1) we deduce $|a+b| \leq |a| + |b|$. (8 points)

3. (20 points total) Let $\epsilon > 0$. We wish to find $\delta > 0$ so that $0 < |x-a| < \delta$ implies $|f(x) - L| < \epsilon$, where $L$ is given below.

a) $f(x) = c$ for all $x \in \mathbb{R}$ and $L = c$. Therefore

$$|f(x) - L| = |c-c| = 0 < \epsilon$$

for all $x \in \mathbb{R}$. Thus any $\delta > 0$ will do. (Calculation 2 points, $\delta$ 3 points)

b) $f(x) = cx + d$ for all $x \in \mathbb{R}$ and $L = ca + d$. If $c = 0$ then $f(x) = d$ is a constant function and

$$\lim_{x \to a} f(x) = d = 0a + d = ca + d$$

by part a). Suppose that $c \neq 0$. Then

$$|f(x) - (ca + d)| = |(cx + d) - (ca + d)| = |c(x-a)| = |c||x-a| < \epsilon;$$

the inequality holds if $|x-a| < \frac{\epsilon}{|c|}$. Take $\delta = \frac{\epsilon}{|c|}$. (Calculation 10 points, $\delta$ 5 points)

4. (20 points total) The containment $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$ is equality in some cases and not in general.
a) We are to show that $(A \times B) \cup (A \times D) = (A \cup A) \times (B \cup D)$ which is better written as $(A \times B) \cup (A \times D) = A \times (B \cup D)$. To do this we show that each side of the equation is a subset of the other.

Suppose that $(x, y) \in (A \times B) \cup (A \times D)$. Then $(x, y) \in A \times B$ or $(x, y) \in A \times D$. In the first case $x \in A$ and $y \in B$. In the second $x \in A$ and $y \in D$. Thus $x \in A$ and $y \in B \cup D$ which is to say $(x, y) \in A \times (B \cup D)$. We have shown $(A \times B) \cup (A \times D) \subseteq A \times (B \cup D)$.

Conversely, suppose that $(x, y) \in A \times (B \cup D)$. Then $x \in A$ and $y \in B \cup D$. Therefore $(x, y) \in A \times B$ or $(x, y) \in A \times D$ which means $(x, y) \in (A \times B) \cup (A \times D)$. We have shown that $A \times (B \cup D) \subseteq A \times (B \cup D)$. Therefore $(A \times B) \cup (A \times D) = A \times (B \cup D)$. (7 points)

b) We are to show that $(A \times B) \cup (C \times B) = (A \cup C) \times (B \cup B)$ which is better written as $(A \times B) \cup (C \times B) = (A \cup C) \times B$. To do this we show that each side of the equation is a subset of the other.

Suppose that $(x, y) \in (A \times B) \cup (C \times B)$. Then $(x, y) \in A \times B$ or $(x, y) \in C \times B$. In the first case $x \in A$ and $y \in B$. In the second $x \in C$ and $y \in B$. Thus $x \in A \cup C$ and $y \in B$ which is to say $(x, y) \in (A \cup C) \times B$. We have shown $(A \times B) \cup (C \times B) \subseteq (A \cup C) \times B$.

Conversely, suppose that $(x, y) \in (A \cup C) \times B$. Then $x \in A \cup C$ and $y \in B$. Therefore $(x, y) \in A \times B$ or $(x, y) \in C \times B$ which means $(x, y) \in (A \cup C) \times B$. We have shown that $(A \cup C) \times B \subseteq (A \times B) \cup (C \times B)$. Therefore $(A \times B) \cup (C \times B) = (A \cup C) \times B$. (7 points)

c) Let $A = \{1\}$ and $B = \{2\}$ for example. Then $A \times B = \{(1, 2)\}$ and $B \times A = \{(2, 1)\}$. Therefore

$$A \cup B = \{(1, 2), (2, 1)\}.$$ 

On the other hand, since $A \cup B = B \cup A = \{1, 2\}$, we have

$$(A \cup B) \times (B \cup A) = \{(1, 1), (1, 2), (2, 1), (2, 2)\}.$$ 

Therefore $(A \times B) \cup (B \times A) \subseteq (A \times B) \times (B \cup A)$. (6 points)