Math 215, Fall 05 Homework #7

Due Friday, 10/21/05 at the beginning of class. Problem 3.e) corrected 10/19/05.

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1. This problem is about the proof of Theorem 1 implies Theorem 2 as discussed in class. Regard Theorem 1 as a statement P and Theorem 2 as the statement "Q implies R". Then the statement "Theorem 1 implies Theorem 2" is can be expressed as "P implies (Q implies R)".

a) Show that the statement "P implies (Q implies R)" is logically equivalent to "(P and Q) implies R". [Hint: Show that both statements are false under the same conditions in terms of the truth values of P, Q, and R. Try to avoid writing out the truth tables, which have eight rows.]

b) Show that to establish "Theorem 1 implies Theorem 2" it suffices to show that whenever P and Q are true then R is true.

2. Let A and B be sets. Then $A\times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$. For $a, a' \in A$ and $b, b' \in B$ we have $(a,b) = (a', b')$ if and only if $a = a'$ and $b = b'$. When A and B are finite, the Cartesian product $A\times B$ has $|A\times B| = |A||B|$ elements.

For sets $A_1, \ldots, A_n$ we define $A_1\times \cdots \times A_n$ inductively by

$$A_1\times \cdots \times A_n = \begin{cases} A_1 & : \quad n = 1 \\ (A_1\times\cdots\times A_{n-1})\times A_n & : \quad n > 1 \end{cases}.$$

For $a_1 \in A_1, \ldots, a_n \in A_n$ we define

$$(a_1, \ldots, a_n) = \begin{cases} a_1 & : \quad n = 1 \\ ((a_1, \ldots, a_{n-1}), a_n) & : \quad n > 1 \end{cases}.$$

Let $n \geq 2$ and $A_1, \ldots, A_n$ be sets.

a) Suppose $a_\imath, a'_\imath \in A_\imath$ for all $1 \leq \imath \leq n$. Show, by induction, that $(a_1, \ldots, a_n) = (a'_1, \ldots, a'_n)$ only if $a_\imath = a'_\imath$ for all $1 \leq \imath \leq n$.

b) Suppose that $A_1, \ldots, A_n$ are finite sets. Show, by induction, that $|A_1 \times \cdots \times A_n| = |A_1| \cdots |A_n|$.

3. This problem is about the algebra of composition of functions.

a) Let $f : A \longrightarrow B$, $g : B \longrightarrow C$, and $h : C \longrightarrow D$ be functions. Show that $(h \circ g) \circ f = h \circ (g \circ f)$.

b) Let $f : A \longrightarrow B$ be a function. Show that $I_B \circ f = f = f \circ I_A$.

c) Let $A = \{a, b\}$ be a set with two elements. Find functions $f, g : A \longrightarrow A$ such that $f \circ g \neq g \circ f$.

d) Suppose that A is a non-empty set and $f \circ g = g \circ f$ for all functions $f, g : A \longrightarrow A$. Show that A has one element.

e) Let $f : A \longrightarrow B$ and $g, h : B \longrightarrow A$ be functions. Suppose that $g \circ f = I_A$ and $f \circ h = I_B$. Show that $h = g$.

4. Let $f : A \longrightarrow B$ and $g : B \longrightarrow A$ be functions which satisfy $g \circ f = I_A$ (as is the case in part e) of the preceding problem).

a) Show $\forall a \in A, \exists b \in B, g(b) = a$.

b) Show $\forall a \in A, \forall a' \in A, f(a) = f(a')$ implies $a = a'$.

c) Write out in words an equivalent of the statement of part b), where the contrapositive of the predicate is used.