Math 215, Fall 05  Homework #8
Solution
11/03/05

1. (20 points total) Here we show, in particular that there is a bijective correspondence between the set of all subsets of a (universal) non-empty set \( U \) and the set of all functions \( f : U \rightarrow \mathbb{R} \) with \( \text{Im} f \subseteq \{0, 1\} \).

a) Suppose that \( A = B \). Then \( \chi_A = \chi_B \). (2 points) Conversely, suppose that \( \chi_A = \chi_B \). We will show that \( A = B \). Let \( x \in A \). Then \( 1 = \chi_A(x) = \chi_B(x) \) which means that \( x \in B \) (otherwise \( \chi_B(x) = 0 \)). Therefore \( x \in B \).

We have shown that \( \chi_A = \chi_B \) implies \( A \subseteq B \). Since we are assuming \( \chi_A = \chi_B \), we have \( A \subseteq B \) and \( \chi_B = \chi_A \). By our previous argument \( B \subseteq A \). Thus \( A = B \). (4 points)

b) First \( \chi_\emptyset = 0 \). Let \( x \in U \). Then \( x \notin \emptyset \). Therefore \( \chi_\emptyset(x) = 0 \). We have shown \( \chi_\emptyset(x) = 0 \) for all \( x \in U \). Therefore \( \chi_\emptyset = 0 \). (3 points)

Next \( \chi_U = 1 \). For all \( x \in U \) we have \( \chi_U(x) = 1 \) by definition of the characteristic function. Therefore \( \chi_U = 1 \). (3 points)

c) Let \( A = \{ x \in U \mid f(x) = 1 \} \). Let \( x \in U \). If \( x \in A \) then \( f(x) = 1 \) by definition of \( A \). Suppose \( x \notin A \). Then \( f(x) \neq 1 \) by definition of \( A \). Since \( \text{Im} f \subseteq \{0, 1\} \) and \( f(x) \neq 1 \) necessarily \( f(x) = 0 \). Thus \( f(x) = \chi_A(x) \) for all \( x \in U \) which means that \( f = \chi_A \). (8 points)

2. (20 points total) \( \chi_{A^c} \) and \( \chi_{A \cap B} \) are algebraic combinations of characteristic functions \( (1 = \chi_U) \).

a) Let \( x \in U \). Since \( x \in A \) if and only if \( x \notin A^c \), and thus \( x \in A^c \) if and only if \( x \notin A \), the calculation

\[
(1 - \chi_A)(x) = \begin{cases} 
1 - 1 : x \in A; \\
1 - 0 : x \notin A
\end{cases} = \begin{cases} 
0 : x \notin A^c; \\
1 : x \in A^c
\end{cases}
\]

shows that \( (1 - \chi_A)(x) = \chi_{A^c}(x) \) for all \( x \in U \). Therefore \( 1 - \chi_A = \chi_{A^c} \). (6 points)
b) Let \( x \in U \). Then
\[
(\chi_A \chi_B)(x) = \chi_A(x) \chi_B(x) = \begin{cases} 
1 \cdot 1 : & x \in A, x \in B; \\
1 \cdot 0 : & x \in A, x \notin B; \\
0 \cdot 1 : & x \notin A, x \in B; \\
0 \cdot 0 : & x \notin A, x \notin B. 
\end{cases}
\]
Since \( x \in A \cap B \) if and only if \( x \in A \) and \( x \in B \), we conclude that
\[
(\chi_A \chi_B)(x) = \begin{cases} 
1 : & x \in A \cap B; \\
0 : & x \notin A \cap B
\end{cases}
= \chi_{A \cap B}(x)
\]
for all \( x \in U \). Thus \( \chi_A \chi_B = \chi_{A \cap B} \). (6 points)

c) Here is a solution which uses parts a) and b).
\[
\chi_{A \cup B} = \chi_{(A' \cap B')^c} \\
= 1 - \chi_{A' \cap B'} \\
= 1 - \chi_{A^c} \chi_{B^c} \\
= 1 - (1 - \chi_A)(1 - \chi_B) \\
= 1 - (1 - \chi_A - \chi_B + \chi_A \chi_B) \\
= 1 + \chi_A + \chi_B - \chi_A \chi_B \\
= \chi_A + \chi_B - \chi_A \chi_B.
\]
Thus \( \chi_{A \cup B} = \chi_A + \chi_B - \chi_A \chi_B \). (4 points)

Note that \( \chi_{A \cup B} = \chi_{A \cap B} = \chi_{A B} = \chi_A(1 - \chi_B) \) so \( \chi_{A \cup B} = \chi_A(1 - \chi_B) \). (4 points)

3. (20 points total) \( f : A \rightarrow B \) and \( g : B \rightarrow C \) are functions.

a) Suppose that \( f, g \) are injective and let \( a, a' \in A \) satisfy \((g \circ f)(a) = (g \circ f)(a')\). Then \( g(f(a)) = g(f(a')) \) by definition of function composition. Since \( g \) is injective \( f(a) = f(a') \). Since \( f \) is injective \( a = a' \). We have shown that \( g \circ f \) is injective. (6 points)

b) Suppose that \( f, g \) are surjective and let \( c \in C \). Since \( g \) is surjective \( c = g(b) \) for some \( b \in B \). Since \( f \) is surjective \( b = f(a) \) for some \( a \in A \). Therefore \((g \circ f)(a) = g(f(a)) = g(b) = c \). We have shown that \( g \circ f \) is surjective. (6 points)
c) We use Problem 3 of Homework 7 in calculating
\[
(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ (g \circ f)) \\
= f^{-1} \circ ((g^{-1} \circ g) \circ f) \\
= f^{-1} \circ (I_B \circ f) \\
= f^{-1} \circ f \\
= I_A
\]
and
\[
(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ (f^{-1} \circ g^{-1})) \\
= g \circ ((f \circ f^{-1}) \circ g^{-1}) \\
= g \circ (I_B \circ g^{-1}) \\
= g \circ g^{-1} \\
= I_C.
\]

(8 points)

4. (20 points total) We complete the square and find that
\[
f(x) = x^2 + 3x - 4 = [(x + \frac{3}{2})^2 - \frac{9}{4}] - 4 = (x + \frac{3}{2})^2 - \frac{25}{4} \geq \frac{-25}{4}.
\]

a) For example \(f(-1) = f(-2) = -6\); therefore \(f\) is not injective. (5 points)
b) Since \(f(x) \geq \frac{-25}{4}\) for all \(x \in \mathbb{R}\), \(f\) is not surjective. Specifically: \(c = -7 < \frac{-25}{4}\). Then \(f(x) \neq c\) for all \(x \in \mathbb{R}\). (5 points)
c) By definition
\[
f^{-1}(\{0, 75/4\}) = \{x \in \mathbb{R} | f(x) \in \{0, 75/4\}\} \\
= \{x \in \mathbb{R} | f(x) = 0 \text{ or } f(x) = 75/4\}.
\]
Since \(f(x) = 0\) if and only if \(f(x) = (x + \frac{3}{2})^2 - \frac{25}{4} = 0\) if and only if \((x + \frac{3}{2})^2 = \frac{25}{4}\) if and only if \(x + \frac{3}{2} = \pm \frac{5}{2}\) if and only if \(x = 1\) or \(x = -4\), and since likewise
\( f(x) = \frac{75}{4} \) if and only if \( x = \frac{7}{2} \) or \( x = -\frac{13}{2} \), it follows that

\[
f^{-1}(\{0, 75/4\}) = \{-4, 1, 7/2, -13/2\}. \quad (5 \text{ points})
\]

d) It is not difficult to show that \( f(x) \) is an increasing function on the interval \([0, 1]\). Since \( f([0, 1]) \) is an interval, \( f([0, 1]) = [f(0), f(1)] = [-4, 0] \). (5 points)