Math 215, Fall 05 Homework #9 Solution

11/08/05

1. (**20 points total**) Here we have examples of induction arguments which come down to the case of $n = 2$.

a) We need to establish the following assertion by induction on $n \geq 1$: Suppose that $X_1, \ldots, X_n, A$ are sets such that $X_1 \cap A = \cdots = X_n \cap A = \emptyset$. Then $(X_1 \cup \cdots \cup X_n) \cap A = \emptyset$. The assertion is true for $n = 1$ as this case boils down to $X_1 \cap A = \emptyset$ which is true by assumption.

Suppose that the assertion is true for $n \geq 1$. Let $X_1, \ldots, X_{n+1}, A$ be sets such that $X_1 \cap A = \cdots = X_{n+1} \cap A = \emptyset$. Then $X_1 \cap A = \cdots = X_n \cap A = \emptyset$ which means $(X_1 \cup \cdots \cup X_n) \cap A = \emptyset$ by our induction hypothesis. Using Theorem 6.3.4(iii) we have

$$\begin{aligned} (X_1 \cap \cdots X_{n+1}) \cap A &= ((X_1 \cap \cdots X_n) \cap X_{n+1}) \cap A \\ &= ((X_1 \cap \cdots X_n) \cap A) \cup (X_{n+1} \cap A) \\ &= \emptyset \cap \emptyset \\ &= \emptyset. \end{aligned}$$

We have shown that if the assertion holds for $n \geq 1$ then it holds for $n + 1$. Thus the assertion holds for all $n \geq 1$ by induction. (**10 points**)

b) We need to establish the following assertion by induction on $n \geq 1$: Suppose that $X_1, \ldots, X_n$ are finite pairwise disjoint sets. Then $X_1 \cup \cdots \cup X_n$ is finite and $|X_1 \cup \cdots \cup X_n| = |X_1| + \cdots + |X_n|$.

The assertion is true for $n = 1$ as this case boils down to X_1 is finite, which holds by assumption, and $|X_1| = |X_1|$.

Suppose that the assertion holds for $n \geq 1$. Let $X_1, \ldots, X_{n+1}$ be finite pairwise disjoint sets. Then $X_1, \ldots, X_n$ are finite pairwise disjoint sets. By the induction hypothesis $X_1 \cup \cdots \cup X_n$ is finite and

$$|X_1 \cup \cdots \cup X_n| = |X_1| + \cdots + |X_n|. \tag{1}$$

Now $X_1 \cup \cdots \cup X_n$ and X_{n+1} are disjoint sets by part a). Thus by Proposition 10.2.1 and (1) we have

$$X_1 \cup \cdots \cup X_{n+1} = (X_1 \cup \cdots \cup X_n) \cup X_{n+1}$$

is finite and

$$|X_1 \cup \cdots \cup X_{n+1}| = |(X_1 \cup \cdots \cup X_n)| + |X_{n+1}| = (|X_1| + \cdots + |X_n)|) + |X_{n+1}|.$$

We have shown that if the assertion holds for $n \geq 1$ then it holds for $n+1$. Thus the assertion holds for all $n \geq 1$ by induction. (**10 points**)

2. (**20 points total**) First of all, suppose that X, Y are *any* sets and $X \subseteq Y$. Then

$$Y = X \cup (Y - X) \qquad \text{and} \qquad X \cap (Y - X) = \emptyset. \tag{2}$$

Since $y \in Y - X$ implies $y \notin X$ by definition, $X \cap (Y - X) = \emptyset$.

We show that $Y = X \cup (Y - X)$. Suppose $y \in Y$. Then $y \in X$ or $y \notin X$, In the first case $y \in X \cap Y$ and in the second $y \in Y - X$. In any event $y \in X \cup (Y - X)$. (We have shown that $Y \subseteq X \cup (Y - X)$ for any sets X, Y.)

Conversely, suppose $y \in X \cup (Y - X)$. Then $x \in X$ or $y \in Y - X$. In the first case $y \in Y$ since $X \subseteq Y$ by assumption. In the second $y \in Y$ by definition. Therefore $y \in Y$ in any case.

a) We are assuming that X, Y are finite sets, $X \subseteq Y$ and $|X| = |Y|$. By Theorem 10.3.1 and (2) we have

$$|X| = |Y| = |X| + |Y - X|$$

from which we deduce that $|Y - X| = 0$. Therefore $Y - X = \emptyset$ which means $Y = X \cup (Y - X) = X \cup \emptyset = X$. (**10 points**)

b) We are assuming that X, Y are finite sets and $|X \cap Y| = |X \cup Y|$. Note that

$$X \cap Y \subseteq X \cup Y.$$

To see this, suppose $x \in X \cap Y$. Then $x \in X$ which means that $x \in X \cup Y$. (Note that finiteness is not used here.)

Since $X \cap Y$ and $X \cup Y$ are finite, we can use part a) to deduce that $X \cap Y = X \cup Y$. We will show that

$$X \cap Y = X \cup Y \qquad \text{implies} \qquad X \subseteq Y. \tag{3}$$

Suppose that $x \in X \cup Y$. Then $x \in X \cup Y$ by definition. Since $X \cup Y = X \cap Y$ by assumption, $x \in X \cap Y$ which means that $x \in Y$ by definition of intersection.

We have shown that $X \cap Y = X \cup Y$ implies $X \subseteq Y$. As $Y \cap X = X \cap Y$ and $Y \cup X = X \cup Y$, by the same result $X \cap Y = X \cup Y$ implies $Y \subseteq X$. Therefore $X \cap Y = X \cup Y$ implies $X = Y$. The proof of part b) is complete. (**10 points**)

3. (**20 points total**) Parts a) and b) are straightforward applications of the inclusion-exclusion principle and part c) involves another idea.

a) $|X \cup Y| = |X| + |Y| - |X \cap Y| = 11 + 6 - 4 = 13$. (**4 points**)

b) $|X^c \cup Y^c| = |(X \cap Y)^c| = |U| - |X \cap Y| = 21 - |X \cap Y|$. Since

$$11 = |X \cup Y| = |X| + |Y| - |X \cap Y| = 4 + 10 - |X \cap Y|$$

we have $|X \cap Y| = 3$. Therefore $|X^c \cup Y^c| = 18$. (**4 points**)

c) Suppose that U is a universal set and $X, Y \subseteq U$. Then

$$X = (X \cap Y) \cup (X \cap Y^c) \qquad \text{and is a disjoint union.} \tag{4}$$

That the union is disjoint follows from $Y \cap Y^c = \emptyset$. To see the equation, suppose that $x \in X$. Then $x \in Y$ in which case $x \in X \cap Y$, or $x \notin Y$, in which case $x \in X \cap Y^c$. Therefore $x \in (X \cap Y) \cup (X \cap Y^c)$. Conversely, suppose $x \in (X \cap Y) \cup (X \cap Y^c)$. Then $x \in X \cap Y$, in which case $x \in X$, or $x \in X \cap Y^c$, in which case $x \in X$. In any event $x \in X$.

By the inclusion-exclusion principle, or (4), we have $|U| = |S \cup S^c| = |S| + |S^c| = |S| + |T|$, and likewise $|U| = |B| + |R|$. Therefore $|T| = 7$ and $|B| = 8$.

(1) $|S \cup B| = |S| + |B| - |S \cap B| = 12 + 8 - 4 = 16$. (**4 points**)

(2) Using (2) we calculate

$$8 = |B| = |B \cap S| + |B \cap S^c| = |B \cap S| + |B \cap T| = 4 + |B \cap T|$$

from which we deduce that $|B \cap T| = 4$. Since

$$7 = |T| = |T \cap R| + |T \cap R^c| = |T \cap R| + |T \cap B| = |T \cap R| + 4$$

we have $|T \cap R| = 3$. (**4 points**)

(3) Using the calculations of part (2), we have $|B^c \cup T^c| = |(B \cap T)^c| = |U| - |B \cap T| = 19 - 4 = 15$. (**4 points**)

4. (**20 points total**) There are six types of currency listed. Thus a person may have $0, 1, \ldots,$ or 6 of the types.

Let $f : X \longrightarrow \{0, 1, \ldots, 6\} = Y$ be the function defined by $f(x)$ is the number of types of these currencies person $x \in X$ has. Since $|Y| = 7$, when $|X| \geq 8$ it follows by the pigeonhole principle that there are distinct $x, x' \in X$ such that $f(x) = f(x')$. By definition of f the two persons x and x' have the same number of types of currencies listed.