Quiz 5 (Version I)

Math 220

Name (print) __________________________

Discussion (circle day, time) Tu Th 10 12

(1) Show your work for full credit. (2) Give exact answers whenever possible; otherwise give answers accurate to two decimal places. (3) You are expected to abide by the University’s rules concerning academic honesty.

1. (10 pts.) Determine the Taylor polynomial $p_4(x)$ approximating the solution at $x_0 = 0$ of the initial value problem $y'' = (x + 2)^3 + y^2$, where $y(0) = 1$ and $y'(0) = -2$.

Solution: The equation is

$$y'' = (x + 2)^3 + y^2.$$  

Therefore

$$y''' = 3(x + 2)^2 + 2yy';$$
$$y'' = 6(x + 2) + 2y'y' + y'y''.$$  

Since $y(0) = 1$ and $y'(0) = -2$ we have

$$y''(0) = 2^3 + (-1)^2 = 9;$$
$$y'''(0) = 3(2^2) + 2(1)(-2) = 8;$$
$$y'(0) = 6(2) + 2((-2)^2 + (1)(9)) = 38.$$  

Therefore

$$p_4(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y'(0)(0)}{4!}x^4 = 1 - 2x + \frac{9}{2}x^2 + \frac{4}{3}x^3 + \frac{19}{12}x^4.$$  

(4 points)

2. (10 pts.) Suppose that $y = \sum_{n=0}^{\infty} a_n x^n$ is a power series solution to $y' + 5x = 2y$. Express $a_1, a_2,$ and $a_3$ in terms of $a_0$.

Solution: Write $y = \sum_{n=0}^{\infty} a_n x^n$. Thus $y' = \sum_{n=0}^{\infty} (n + 1) a_{n+1} x^n$. The equation is

$$\sum_{n=0}^{\infty} (n + 1) a_{n+1} x^n + 5x = \sum_{n=0}^{\infty} 2a_n x^n.$$  

(3 points)

Comparing coefficients we have

$$a_1 = 2a_0;$$
$$2a_2 + 5 = 2a_1;$$
$$3a_3 = 2a_2.$$  

Therefore

$$a_2 = \frac{1}{2} (2a_1 - 5) = 2a_0 - \frac{5}{2};$$
$$a_3 = \frac{2}{3} a_2 = \frac{4}{3} a_0 - \frac{5}{3}.$$  

(7 points)

1
1. (10 pts.) Determine the Taylor polynomial $p_4(x)$ approximating the solution at $x_0 = 0$ of the initial value problem $y'' = (x + 1)^3 + y^2$, where $y(0) = 3$ and $y'(0) = -1$.

**Solution:** The equation is
\[ y'' = (x + 1)^3 + y^2. \]

Therefore
\[
\begin{align*}
  y^{(3)}(0) & = 3(x + 1)^2 + 2yy(1); \\
  y^{(4)}(0) & = 6(x + 1) + 2(y^{(3)}(1) + yy(2)).
\end{align*}
\]

Since $y(0) = 3$ and $y^{(1)}(0) = -1$ we have
\[
\begin{align*}
  y^{(2)}(0) & = 1^3 + 3^2 = 10; \\
  y^{(3)}(0) & = 3(1^2) + 2(3)(-1) = -3; \\
  y^{(4)}(0) & = 6(1) + 2((-1)^2 + 3(10)) = 68.
\end{align*}
\]

Therefore
\[
p_4(x) = y(0) + y^{(1)}x + \frac{y^{(2)}(0)}{2!}x^2 + \frac{y^{(3)}(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 = 3 - x + 5x^2 - \frac{1}{2}x^3 + \frac{17}{6}x^4.
\]

2. (10 pts.) Suppose that $y = \sum_{n=0}^{\infty} a_n x^n$ is a power series solution to $y' + 3x = 4y$. Express $a_1, a_2, a_3$ in terms of $a_0$.

**Solution:** Write $y = \sum_{n=0}^{\infty} a_n x^n$. Thus $y' = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n$. The equation is
\[
\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + 3x = \sum_{n=0}^{\infty} 4a_n x^n.
\]

Comparing coefficients we have
\[
\begin{align*}
a_1 & = 4a_0; \\
2a_2 + 3 & = 4a_1; \\
3a_3 & = 4a_2.
\end{align*}
\]

Therefore
\[
\begin{align*}
a_2 & = \frac{1}{2} (4a_1 - 3) = 8a_0 - \frac{3}{2}; \\
a_3 & = \frac{4}{3} a_2 = \frac{32}{3} a_0 - 2.
\end{align*}
\]