1. (20 pts.) Find the Fourier cosine series of \(f(x) = x\) on \([0, 1]\).

\[ \text{Solution: Since } L = 1 \text{ we have } \frac{n\pi}{L} = n\pi. \text{ Thus the series has the form} \]

\[ \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x. \textbf{ (7 points)} \]

Now

\[ a_0 = \frac{2}{L} \int_{0}^{1} x \, dx = 2 \int_{0}^{1} x \, dx = 1 \]

and for \(n > 0\), with the substitution \(u = n\pi x\), we calculate

\[ a_n = \frac{2}{L} \int_{0}^{1} x \cos n\pi x \, dx \]

\[ = \frac{2}{n^2\pi^2} \int_{0}^{n\pi} u \cos u \, du \]

\[ = \frac{2}{n^2\pi^2} [u \sin u + \cos u]_{0}^{n\pi} \]

\[ = \frac{2}{n^2\pi^2} (\cos n\pi - 1) \]

\[ = \frac{2}{n^2\pi^2} ((-1)^n - 1). \]

Thus

\[ \frac{a_0}{2} = \frac{1}{2} \text{ and } a_n = \frac{2}{n^2\pi^2}(\cos n\pi - 1) = \frac{2}{n^2\pi^2}((-1)^n - 1) \text{ for } n > 0. \]

and hence

\[ f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}((-1)^n - 1) \cos n\pi x. \textbf{ (13 points)} \]
1. (20 pts.) Find the Fourier sine series of \( f(x) = x \) on \([0, 1] \).

**Solution:** Since \( L = 1 \) we have \( \frac{n\pi}{L} = n\pi \). Thus the series has the form

\[
\sum_{n=1}^{\infty} b_n \sin(n\pi x). \quad (7 \text{ points})
\]

With the substitution \( u = n\pi x \), we calculate

\[
b_n = \frac{2}{L} \int_{0}^{1} x \sin (n\pi x) \, dx
\]

\[
= \frac{2}{n^2\pi^2} \int_{0}^{\pi} u \sin u \, du
\]

\[
= \frac{2}{n^2\pi^2} \left[ -u \cos u + \sin u \right]_{0}^{\pi}
\]

\[
= \frac{2}{n^2\pi^2} (-n\pi \cos n\pi)
\]

\[
= \frac{2}{n\pi} (-(-1)^n).
\]

Thus \( b_n = \frac{2}{n\pi} (-(-1)^n) = \frac{2}{n\pi} (-1)^{n+1} \) for \( n \geq 1 \).

and hence

\[
f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi} (-1)^{n+1} \sin(n\pi x). \quad (13 \text{ points})
\]