Homework #10 (week of 10/25–10/29)

Due Friday, 10/29/04 in class

1. Let $G$ be a group and $H$ be a subgroup of $G$.
   a) If $|G:H| = 2$ show that $H$ is a normal subgroup of $G$.
   b) Suppose that $|H| = 2$. Show that $H$ is a normal subgroup of $G$ if and only if $H \subseteq Z(G)$.
   c) Determine all normal subgroups of $G = S_3$.

2. Let $G = \text{GL}(n, \mathbb{R})$ be the group of all $n \times n$ invertible matrices with real coefficients under matrix multiplication. Suppose that $H$ is a subgroup of $G$. Show that
   
   $$N = \{ h \in H \mid \text{Det}(h) = 1 \}$$

   is a normal subgroup of $H$.

3. Let $G$ be a group and suppose that $\{H_i\}_{i \in I}$ is an indexed family of subgroups of $G$. Then $H = \cap_{i \in I} H_i$ is a subgroup of $G$.
   Suppose that $H_i$ is a normal subgroup of $G$ for all $i \in I$. Show that $H$ is a normal subgroup of $G$.

4. Let $G = S_n$ and $\sigma = (a_1 \ a_2 \ \cdots \ a_m) \in G$ be an m-cycle. Then for $\tau \in G$ the element $\tau \sigma \tau^{-1}$ is an m-cycle given by
   
   $$\tau \sigma \tau^{-1} = (\tau(a_1) \ \tau(a_2) \ \cdots \ \tau(a_m)).$$  \hspace{1cm} (1)

   Use (1) to find all normal subgroups of $G = S_4$. 

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