Solution to Homework # 10 (week of 10/25–10/29)

10/29/04 Radford

1. (9 points total) a) Suppose that $|G:H| = 2$. Then $H$ has two distinct left cosets in $G$. Since the distinct left cosets of $H$ in $G$ partition $G$, and $H$ is a left coset, it follows that the other must be $G\setminus H$. Right cosets of $H$ in $G$ partition $G$ as well, and the number must be $|G|/|H| = 2$. Since $H$ is a right coset of $H$ in $G$ the other must be $G\setminus H$. Since $\{H, G\setminus H\}$ is the set of left cosets and the set of right cosets of $H$ in $G$, it follows that $H$ is a normal subgroup of $G$. (3 points)

b) We are assuming that $|H| = 2$. Suppose, first of all, that $H \subseteq Z(G)$. Then $aH = \{ah \mid h \in H\} = \{ha \mid h \in H\} = Ha$ for all $a \in G$ since $h \in H$ is in the center of $G$. Therefore $H$ is a normal subgroup of $G$.

Conversely, suppose that $H$ is a normal subgroup of $G$ and let $a \in G$. Since $|H| = 2$ we may write $H = \{e, h\}$. Since $H$ is normal

$$\{e, h\} = H = aHa^{-1} = \{aea^{-1}, aha^{-1}\} = \{e, aha^{-1}\}.$$ 

Since $h \neq e$ it follows that $h = aha^{-1}$. Thus $ha = (aha^{-1})a = ahe = ah$ (associativity details have been omitted). We have shown that $h \in Z(G)$. Since $Z(G)$ is a subgroup of $G$ it follows that $e \in Z(G)$. Therefore $H \subseteq Z(G)$. (3 points)

c) $G = S_3$ and $(e)$ are normal subgroups of $G$. Since $|S_3:A_3| = |S_3|/|A_3| = 6/3 = 2$, the subgroup $A_3$ is a normal subgroup of $G$ by part a).

Suppose that $H$ is a normal subgroup of $G$ other than these three. Now $|H| = 1, 2, 3, \text{ or } 6$ by Lagrange’s theorem. Thus $|H| = 2$ or 3. If $|H| = 3$ then $H$ is cyclic of order 3. In this case $H$ contains a 3-cycle and thus $H = A_3$. Therefore $|H| = 2$.

Since $H$ has prime order 2 it follows that $H$ is cyclic. Thus $H = \{\text{Id}, (ab)\}$ where the latter is some 2-cycle. We may write $\{1, 2, 3\} = \{a, b, c\}$. Since $(ac)(ab) = (abc) \neq (acb) = (ab)(ac)$ it follows that $(ab) \notin Z(G)$. Therefore $H$ is not a normal subgroup of $G$ by part b).

We have shown that the normal subgroups of $G = S_3$ are $G$, $\{\text{Id}\}$, and $A_3$. (3 points for analysis and answer)
2. \textbf{(10 points total)} Recall that $\det(I_n) = 1$ and $\det(h\ell) = \det(h)\det(\ell)$ for all $h, \ell \in \text{GL}(n, \mathbb{R})$. As a consequence $\det(h) \neq 0$ for all $h \in \text{GL}(n, \mathbb{R})$ and $\det(h^{-1}) = (\det(h))^{-1}$. These are results of linear algebra which we basically assume.

First of all $N$ is a subgroup of $H$. For $\det(I_n) = 1$ shows that the $n \times n$ identity matrix belongs to $N$. Thus $H \neq \emptyset$. \textbf{(3 points)}

Let $h, \ell \in N$. Then $h, \ell \in H$ so $h\ell^{-1} \in H$ be the 1-Step Subgroup Test. The calculation

$$\det(h\ell^{-1}) = \det(h)\det(\ell^{-1}) = \det(h)(\det(\ell))^{-1} = 1(1^{-1}) = 1$$

completes the argument that $h\ell^{-1} \in N$. Therefore $N$ is a subgroup of $G$ by the 1-Step Subgroup Test. \textbf{(3 points)}

Suppose that $h \in N$ and $a \in H$. Then $a^{-1} \in H$ so $aha^{-1} \in H$. Since

$$\det(aha^{-1}) = \det(a)\det(h)(\det(a))^{-1} = \det(a)1(\det(a))^{-1} = 1$$

we conclude that $\det(aha^{-1}) = 1$. Therefore $aha^{-1} \in N$. We have shown that $aNa^{-1} \subseteq N$. Therefore $N$ is a normal subgroup of $G$. \textbf{(4 points)}

3. \textbf{(5 points total)} Let $h \in H$, $a \in G$, and $\iota \in I$. Then $h \in H_i$ by definition of intersection. Since $H_i$ is a normal subgroup of $G$ it follows that $aha^{-1} \in H_i$. Therefore $aha^{-1} \in H$. We have shown that $aHa^{-1} \subseteq H$. Therefore $H$ is a normal subgroup of $G$.

4. \textbf{(16 points total)} A permutation of $G = S_4$ which is not the identity is a 2-cycle, 3-cycle, 4-cycle, or the product of two 2-cycles. A group $G$ and its trivial subgroup are always normal subgroups of $G$. Thus $G$ and $\{\text{Id}\}$ are normal subgroups of $G = S_4$. Since $A_4$ has index 2 in $G$, it follows that $A_4$ is a normal subgroup of $G$ by part a) of Problem 1.

Let $H$ be a normal subgroup of $G$ which is not one of these three. Then $|H| = 2, 3, 4, 6$ or 12. Let $\{a, b, c, d\} = \{1, 2, 3, 4\}$.

\textit{Case 1: $H$ contains a 2-cycle $(ab)$}. Let $\tau = (bc)$. Then the calculation

$$\tau(ab)\tau^{-1} = (ac)$$

shows that all 2-cycles of the form $(ax) \in H$, where $x$ is any of $b, c, d$. Since $ax = (xa)$ the preceding conclusion implies that $(xy) \in H$, where $x, y \in \{1, 2, 3, 4\}$ are different. Therefore all 2-cycles belong to $H$. Since
every permutation is a product of 2-cycles, \( G \subseteq H \). This means \( H = G \), a contradiction. Our conclusion: \( H \) does not contain a 2-cycle.

Case 2: \( H \) contains a 4-cycle \((a\, b\, c\, d)\). Let \( \tau = (b\, c) \). Then

\[
\tau(a\, b\, c\, d)^2 \tau^{-1}(a\, b\, c\, d) = \tau(a\, c)(b\, d)\tau^{-1}(a\, b\, c\, d) = (a\, b)(c\, d)(a\, b\, c\, d) = (a)(b\, d)(c)
\]

is a 2-cycle in \( H \), a contradiction. Our conclusion: \( H \) does not contain a 4-cycle.

From our first two cases we conclude that \( H \) contains no odd permutations. Therefore

\( H \subseteq A_4 \).

Since \(|A_4| = 12\) it follows that \(|H| \) divides 12.

Case 3: \( H \) contains a 3-cycle \((a\, b\, c)\). Let \( \tau = (a\, b\, d) \). Then

\[
\tau(a\, b\, c)\tau^{-1}(a\, b\, c)^{-1} = (b\, d\, c)(a\, c\, b) = (a\, b)(c\, d) \in H.
\]

Since \( H \) has an element of order 3 and an element of order 2 it follows that 6 divides \(|H|\). Let \( K = \langle a\, b\, c \rangle \). Since \(|H| \) divides 12 either \(|H| = 6\) or \(|H| = 12\). In the first case \( K \) is a normal subgroup of \( H \) by part a) of Problem 1. But

\[
(a\, b)(c\, d)(a\, b\, c)(a\, b)(c\, d) = (a\, d\, b)(c) \notin H.
\]

Therefore \( K \) is not a normal subgroup of \( H \). Thus \( H = A_4 \), a contradiction. Our conclusion: \( H \) does not contain a 3-cycle.

Case 4: \( H \) contains a product of two disjoint 2-cycles \((a\, b)(c\, d)\). By the first three cases \( H \) can only contain products of two disjoint 2-cycles and \( \text{Id} \). Let \( \tau = (b\, c) \). Then

\[
\tau(a\, b)(c\, d)\tau^{-1} = \tau(a\, b)\tau^{-1}(c\, d)\tau^{-1} = (a\, c)(b\, d) \in H.
\]

Since \((a\, b)(c\, d) = (a\, b)(d\, c)\) the preceding calculation shows that \((a\, d)(b\, c) \in H\) as well. Thus \( H \) contains all products of disjoint two 2-cycles. Necessarily

\[
H = \{\text{Id}, (1\, 2)(3\, 4), (1\, 3)(2\, 4), (1\, 4)(2\, 3)\}.
\]
The calculations
\[
\left( (ab)(cd) \right) \left( (ac)(bd) \right) = (ad)(bc)
\]
and
\[
\left( (ab)(cd) \right)^2 = \text{Id}
\]
show that $H$ is indeed a subgroup of $G$. Let $\tau \in G$. Then
\[
\tau (ab)(cd)\tau^{-1} = \tau (ab)\tau^{-1}\tau (cd)\tau^{-1} = (\tau (a) \tau (b))(\tau (c) \tau (d))
\]
is the product of two disjoint 2-cycles and thus in $H$. Therefore $H$ is a normal subgroup of $G$. (8 points for the analysis)

To summarize: The normal subgroups of $G = S_4$ are

\begin{align*}
S_4, & \quad (2 \text{ points}) \\
A_4, & \quad (2 \text{ points}) \\
V = \{ \text{Id, (1 2)(3 4), (1 3)(2 4), (1 4)(2 3)} \}, & \quad (2 \text{ points}) \\
\{ \text{Id} \} & \quad (2 \text{ points}).
\end{align*}