Homework #11 (week of 11/01–11/05)

Due Friday, 11/05/04 in class

1. Let $m, n$ positive integers and $G = S_n$. Find all group homomorphisms $f : G \rightarrow \mathbb{Z}_m$. [Hint: Let $f : G \rightarrow \overline{G}$ be a homomorphism of groups. Suppose that $a \in G$ and $a = a_1 \cdots a_r$ where $a_1, \ldots, a_r \in G$. Then $f(a) = f(a_1) \cdots f(a_r) = f(a_1) \cdots f(a_r)$. Recall that all permutations are products of 2-cycles when $n \geq 2$.]

2. Let $G = \langle a \rangle$ and $\overline{G}$ be finite cyclic groups.
   a) Suppose that $f : G \rightarrow \overline{G}$ is an onto homomorphism. Show that $|\overline{G}|$ divides $|G|$ and $f(a)$ generates $\overline{G}$.
   b) Suppose that $|\overline{G}|$ divides $|G|$ and $\overline{G} = \langle b \rangle$. Show that the rule $f : G \rightarrow \overline{G}$ given by $f(a^\ell) = b^\ell$ for all $\ell \in \mathbb{Z}$ is a well-defined onto group homomorphism. (Well-defined means that $a^\ell = a^{\ell'}$ implies $b^\ell = b^{\ell'}$.)
   c) Suppose that $|\overline{G}|$ divides $|G|$ and $\overline{G} = \langle b \rangle$. Use part b) to determine all onto group homomorphisms $f : G \rightarrow \overline{G}$.
   d) Suppose that $G$ has order 30 and $\overline{G} = \langle b \rangle$ has order 20. Use part b) to determine all group homomorphisms, onto or not, $f : G \rightarrow \overline{G}$.

3. Let $R$ be a ring with unity 1 and set $G = \{ r \in R \mid rr' = 1 = r'r \text{ for some } r' \in R \}$.
   a) Show that $G$ is a group under the multiplication in the ring.
   b) Determine $G$ when $R = \text{M}_n(\mathbb{R})$ is the ring of $n \times n$ matrices with real coefficients, where $n \geq 1$.
   c) Determine $G$ when $R = \mathbb{Z}_n$, where $n > 1$. 

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