Solution to Homework #14 (week of 11/22–11/24)

12/02/04 Radford

Suppose that $G$ is an (additive) abelian group and $S_1, \ldots, S_n$ are non-empty subsets of $G$. Then

$$S_1 + \cdots + S_n = \{s_1 + \cdots s_n \mid s_i \in S_i \text{ for all } 1 \leq i \leq n\}.$$

1. (10 points total) $R$ is a ring.
   
a) The indexed family of ideals $\{I_s\}_{s \in S}$ is an indexed family of (additive) subgroups of $R$. We have shown that $I = \cap_{s \in S} I_s$ is an (additive) subgroup of $R$. (3 points)

   Let $r \in R$ and $a \in I$. Then $a \in I_s$ for all $s \in S$. Let $s \in S$. Then $I_s$ is an ideal of $R$. Therefore $ra, ar \in I_s$. Consequently $ra, ar \in \cap_{s \in S} I_s = I$. We have shown that $I$ is an ideal of $R$. (2 points)

   b) Let $I = I_1 + \cdots + I_n$. By assumption $I_1, \ldots, I_n$ are ideals of $R$. Since they are also additive subgroups of $R$ it follows that $I_i \neq \emptyset$ for all $1 \leq i \leq n$. Therefore $I = I_1 + \cdots + I_n \neq \emptyset$. (1 points)

   Let $x, y \in I$. Then $x = a_1 + \cdots + a_n$ and $y = b_1 + \cdots + b_n$ where $a_i, b_i \in I_i$ for all $1 \leq i \leq n$. Let $1 \leq i \leq n$. Since $I_i$ is a additive subgroup $a_i - b_i \in I_i$. Therefore

   $$x - y = (a_1 + \cdots + a_n) - (b_1 + \cdots + b_n) = (a_1 - b_1) + \cdots (a_n - b_n) \in I_1 + \cdots + I_n$$

   which shows that $I$ is an additive subgroup of $R$ by the 1-Step Subgroup Test. (2 points)

   Now let $r \in R$ and let $1 \leq i \leq n$. Since $I_i$ is an ideal of $R$ we have $ra_i, a_i r \in I_i$. Therefore

   $$rx = r(a_1 + \cdots + a_n) = ra_1 + \cdots + ra_n \in I_1 + \cdots + I_n = I$$
and

\[ xr = (a_1 + \cdots + a_n)r = a_1r + \cdots + a_nr \in I_1 + \cdots + I_n = I. \]  (2 points)

This completes our proof that \( I \) is an ideal of \( R \).

2. (10 points total) \( R \) is a ring, \( a \in R \), and \( La = \{ ra \mid r \in L \} \).

a) By assumption \( L \) is a left ideal of \( R \) and \( a \in R \). Since \( L \) is an additive subgroup of \( R \) we have \( L \neq \emptyset \). Thus \( La \neq \emptyset \).

Suppose that \( x, y \in La \). Then \( x = sa \) and \( y = ta \) for some \( s, t \in L \). Since \( L \) is an additive subgroup of \( R \) it follows that \( s - t \in L \) by the 1-Step Subgroup Test. Therefore

\[ x - y = sa - ta = (s - t)a \in La. \]

by the 1-Step Subgroup Test \( La \) is an additive subgroup of \( R \).

Let \( r \in R \). Then

\[ rx = r(sa) = (rs)a \in La \]

since \( L \) is a left ideal of \( R \). Therefore \( La \) is a left ideal of \( R \). (5 points)

b) Suppose that \( L_1, \ldots, L_n \) are left ideals of \( R \). Our argument for part b) of the preceding exercise showed that if \( A_1, \ldots, A_n \) are additive subgroups of \( R \) then \( A_1 + \cdots + A_n \) is an additive subgroup of \( R \). Thus \( L = L_1 + \cdots + L_n \) is an additive subgroup of \( R \).

Suppose that \( r \in R \) and \( x \in L \). Then \( x = a_1 + \cdots + a_n \) where \( a_i \in L_i \) for all \( 1 \leq i \leq n \). Fix \( 1 \leq i \leq n \). Since \( L_i \) is a left ideal of \( R \) and \( a_i \in L_i \) it follows that \( ra_i \in L_i \). Thus

\[ r(a_1 + \cdots + a_n) = ra_1 + \cdots + ra_n \in L_1 + \cdots + L_n = L. \]

We have shown that \( L \) is a left ideal of \( R \). (5 points)

3. (10 points total) Let \( R_1, \ldots, R_n \) be rings with unity and \( R = R_1 \oplus \cdots \oplus R_n \).

a) Since \( I_1, \ldots, I_n \) are ideals of \( R_1, \ldots, R_n \) respectively, they are additive subgroups and hence not empty. Thus \( I_1 \oplus \cdots \oplus I_n \neq \emptyset \). Let \( x, y \in I_1 \oplus \cdots \oplus I_n \).

Then \( x = (a_1, \ldots, a_n) \) and \( y = (b_1, \ldots, b_n) \), where \( a_i, b_i \in I_i \) for all \( 1 \leq i \leq n \). Thus \( a_i - b_i \in I_i \) for all \( 1 \leq i \leq n \) by the 1-Step Subgroup Test. Thus

\[ x - y = (a_1, \ldots, a_n) - (b_1, \ldots, b_n) = (a_1 - b_1, \ldots, a_n - b_n) \in I_1 \oplus \cdots \oplus I_n \]
which means that $I_1 \oplus \cdots \oplus I_n$ is an additive subgroup of $R_1 \oplus \cdots \oplus R_n$ by the 1-Step subgroup Test. Now let $r \in R_1 \oplus \cdots \oplus R_n$. Then $r = (r_1, \ldots, r_n)$ where $r_i \in R_i$ for all $1 \leq i \leq n$. Let $1 \leq i \leq n$. Since $I_i$ is an ideal of $R_i$ the products $r_i a_i, a_i r_i \in R_i$. Therefore

$$rx = (r_1, \ldots, r_n)(a_1, \ldots, a_n) = (r_1 a_1, \ldots, r_n a_n) \in I_1 \oplus \cdots \oplus I_n$$

and

$$xr = (a_1, \ldots, a_n)(r_1, \ldots, r_n) = (a_1 r_1, \ldots, a_n r_n) \in I_1 \oplus \cdots \oplus I_n.$$

This completes our proof that $I_1 \oplus \cdots \oplus I_n$ is an ideal of $R_1 \oplus \cdots \oplus R_n$. (3 points)

b) Let $1 \leq i \leq n$ Then $f_i : R \rightarrow R_i$ defined by $f((a_1, \ldots, a_n)) = a_i$ for all $(a_1, \ldots, a_n) \in R$ is onto. For let $a \in I_i$. Then $f_i((0, \ldots, a_i, \ldots, 0) = a$, where the “a" in the tuple in the $i^{th}$ position.

$$f_i((a_1, \ldots, a_n)(b_1, \ldots, b_n)) = f_i((a_1 b_1, \ldots, a_n b_n))$$

$$= a_i b_i$$

$$= f_i((a_1, \ldots, a_n)) f_i((b_1, \ldots, b_n))$$

for all $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in R$ shows that $f_i$ is a ring homomorphism. (3 points)

c) Suppose that $I$ is an ideal of $R$. Then $f_i(I)$ is an ideal of $R_i$ for all $1 \leq i \leq n$ since $f_i$ is an onto ring homomorphism by part b). Let $x = (a_1, \ldots, a_n) \in R$. Then $f_i(x) = a_i$ which means that $x = (f_1(x), \ldots, f_n(x))$. Therefore $I \subseteq f_1(I) \oplus \cdots \oplus f_n(I)$. Let $J = f_1(I) \oplus \cdots \oplus f_n(I)$. We have shown that $I \subseteq J$. Part c) will follow if $I = J$. To do this we need only show that $J \subseteq I$.

Let $e_i = (0, \ldots, 1, \ldots, 0) \in R$ be the $n$-tuple with entries zero with one exception which has value 1 and is in the $i^{th}$ position. Suppose that $y \in f_1(I) \oplus \cdots \oplus f_n(I)$. Then $y = (f_1(x_1), \ldots, f_n(x_n))$, where $x_i \in I_i$ for all $1 \leq i \leq n$. Thus $x = x_1 e_1 + \cdots + x_n e_n \in I$ since $I$ is an ideal of $R$. Since

$$f_i(x) = f_i(x_1)f_i(e_1) + \cdots + f_i(x_i)f_i(e_i) + \cdots + f_i(x_n)f_i(e_n)$$

$$= f_i(x_1)0 + \cdots + f_i(x_i)1 + \cdots + f_i(x_n)0$$

$$= f_i(x_i)$$
it follows that
\[ x = (f_1(x), \ldots, f_n(x)) = (f_1(x_1), \ldots, f_n(x_n)) = y. \]
Therefore \( y = x \in I \). We have shown that \( J \subseteq I \). (4 points)

4. (10 points total) \( R = \mathbb{Z} \oplus \mathbb{Z} \).

a) Let \( P \) be a prime ideal of \( R \). Since \( R \) is always a prime ideal of \( R \), we may assume that \( P \neq R \). Now \( P = P_1 \oplus P_2 \) for some ideals \( P_1, P_2 \) of \( \mathbb{Z} \), by the preceding exercise. We will show that \( P_1 \) and \( P_2 \) must be prime.

Suppose that \( a, b \in \mathbb{Z} \) and \( ab \in P_1 \). Then
\[ (a, 0)(b, 0) = (ab, 0) \in P_1 \oplus P_2 = P \]
means that \( (a, 0) \in P_1 \) or \( (b, 0) \in P_2 \) since \( P \) is prime. Therefore \( a \in P_1 \) or \( b \in P_2 \). We have shown that \( P_1 \) is prime.

Likewise \( P_2 \) is prime. For suppose that \( ab \in P_2 \). Then
\[ (0, a)(0, b) = (0, ab) \in P_1 \oplus P_2 = P \]
means \( (0, a) \in P_1 \) or \( (0, b) \in P_2 \) since \( P \) is prime. Therefore \( a \in P_1 \) or \( b \in P_2 \).

We show that, in addition, that \( P_1 = \mathbb{Z} \) or \( P_2 = \mathbb{Z} \). For
\[ (1, 0)(0, 1) = (0, 0) \in P \]
implies that \( (1, 0) \in P_1 \) or \( (0, 1) \in P_2 \) since \( P \) is prime. In the first case
\[ (a, 0) = (a, 0)(1, 0) \in P \]
for all \( a \in \mathbb{Z} \) since \( P \) is an ideal of \( R \). Thus \( P_1 = \mathbb{Z} \). In the second case
\[ (0, a) = (0, a)(0, 1) \in P \]
again since \( P \) is an ideal of \( R \). Thus \( P_2 = \mathbb{Z} \).

Suppose that \( Q \) is a prime ideal of \( \mathbb{Z} \). We leave it as a small exercise that the ideals \( Q \oplus \mathbb{Z} \) and \( \mathbb{Z} \oplus Q \) are prime ideals of \( R \). Thus the prime ideals of \( R \) are
\[ Q \oplus \mathbb{Z} \quad \text{and} \quad \mathbb{Z} \oplus Q, \]
where \( Q \) is a prime ideal of \( \mathbb{Z} \). (5 points)

b) Maximal ideals are prime. Thus maximal ideals of \( R \) have the form \( M \oplus \mathbb{Z} \) or \( \mathbb{Z} \oplus M \), where \( M \) is a prime ideal of \( \mathbb{Z} \). Note that any ideal of \( R \) which contains \( M \oplus \mathbb{Z} \) (respectively \( \mathbb{Z} \oplus M \)) must have the form \( M' \oplus \mathbb{Z} \) (respectively \( \mathbb{Z} \oplus M' \)), where \( M' \) is an ideal of \( \mathbb{Z} \). Therefore \( M \) must be maximal. (5 points)