Homework #15 (week of 11/29–12/03)

Due Friday, 12/03/04 in class

1. Let $R = \mathbb{Z}_7[x]$ and $a = x^3 + 3x^2 + 3x + 2$, $b = x^3 + 5x^2 + 2x + 3 \in R$.
   a) Write $a$ and $b$ as a product of monic irreducible factors.
   b) Use the Euclidean algorithm to find the greatest common divisor of $a$ and $b$.
   c) Find a monic polynomial $c \in R$ such that $Rc = Ra + Rb$; the latter is the ideal of $R$ generated by $a$ and $b$.

2. Let $\mathbb{R}$ be the field of real numbers and $R = \mathbb{R}[x]$. In this exercise we construct the field of complex numbers.
   a) Show that $x^2 + 1$ is an irreducible polynomial of $R$. (Hence $R/I$ is a field, where $I = (x^2 + 1)$).
   b) Show that $f : \mathbb{R} \rightarrow R/I$ defined by $f(r) = r + I$ for all $r \in \mathbb{R}$ is a one-one ring homomorphism. (Thus we may think of $\mathbb{R}$ as a subfield of $R/I$ by the identification of $r$ with $r + I$.)
   c) Show that every element of $R/I$ has the form $(r + I) + (s + I)(x + I)$ for some $r, s \in \mathbb{R}$.
   d) Suppose that $r, r', s, s' \in \mathbb{R}$. Show that
      \[(r + I) + (s + I)(x + I) = (r' + I) + (s' + I)(x + I)\]
      implies $r = r'$ and $s = s'$.
   e) Show that $(x + I)^2 = -(1 + I)$. 

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3. Let $R = \mathbb{Z}_2[x]$.

   a) Show that the only irreducible polynomial of degree 2 in $R$ is $x^2 + x + 1$.

Let $I = (x^2 + x + 1)$. Then $I$ is a maximal ideal of $R$ and thus $F = R/I$ is a field.

   b) Show that $F$ has 4 elements and write down the addition table and the multiplication table for $F$.

4. Let $R = \mathbb{Z}_2[x]$. Find all of the irreducible polynomials of degree 3 in $R$. 