Homework # 5 (week of 09/20–09/24)

Due Friday, 09/24/04 in class

Let $G$ be a group. For a non-empty subset $S$ of $G$ the centralizer of $S$ in $G$ is

$$C_G(S) = \{g \in G \mid gs = sg \text{ for all } s \in S\}.$$

1. Let $G$ be a group and suppose that $S$ is a non-empty subset of $G$.
   a) Show that $C_G(S)$ is a subgroup of $G$.
   b) Show that $C_G(S) = \bigcap_{s \in S} C(s)$.
   c) Let $G = \text{GL}(2, \mathbb{R})$ and $S = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{R}\}$. Find $C_G(S)$.

   [Hint: For part c), let $g = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in G$. Then $g \in C_G(S)$ if and only if

   $$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

   for all $a \in \mathbb{R}$. Thus part c) comes down to solving systems of linear equations. It is easy to see that $S$ is a subgroup of $G$; this is not part of the problem.]

2. Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 7 & 6 & 2 & 1 & 3 \end{pmatrix} \in S_7$.
   a) Write each element of $\langle f \rangle$ as a product of disjoint cycles.
   b) Find the order of $f$.
   c) What are the orders of the elements of $S_7$? Justify your answer.
3. Let $G = \mathbb{Z}_{30}$.
   
   a) Describe each subgroup of $G$ by listing its elements.
   
   b) For each order, list the elements of $G$ having that particular order.
   
   c) Find all generators of the subgroup $<3>$ of $G$.
   
   d) Construct a lattice diagram for $G$. 