1. (14 points total) We first note that $308 = 4 \cdot 7 \cdot 11$ and $140 = 4 \cdot 5 \cdot 7$.

a) The greatest common divisor of 308 and 140 is 28. As $<a^{140}> = <a^{\text{gcd}(308,140)}>$, it follows that $d = 28$. (2 points). The order of $H = <a^{28}>$ is therefore $308/28 = 11$ (1 point).

b) $|H| = |<a^{28}>| = 11$. The integers $1 \leq \ell \leq 11$ which are relatively prime to 11 are the ten integers 1, 2, 3, · · · , 10. Therefore the generators of $H = <a^{28}>$ are

\[
(a^{28})^1 = a^{28}, \quad (a^{28})^2 = a^{56}, \quad (a^{28})^3 = a^{84}, \quad (a^{28})^4 = a^{112}, \quad (a^{28})^5 = a^{140},
\]

\[
(a^{28})^6 = a^{168}, \quad (a^{28})^7 = a^{196}, \quad (a^{28})^8 = a^{224}, \quad (a^{28})^9 = a^{252}, \quad (a^{28})^{10} = a^{280}.
\]

(3 points).

c) The divisors of $|H| = 11$ are 1, 11. Thus $H$ has two subgroups; namely $H = <(a^{28})^1> = <a^{28}>$ and $<(a^{28})^{11}> = <a^{308}> = <a^0>$. (3 points).

d) (1 point) and (4 points) respectively. The divisors of 11 are 1, 11. The divisors of 308 are 1, 2, 5, 7, 4, 14, 22, 77, 28, 44, 154, 308. Lines filled in later.

$$<a>$$

$$<a^2> \quad <a^7> \quad <a^{11}>$$

$$<a^4> \quad <a^{14}> \quad <a^{22}> \quad <a^{77}>$$

$$H$$

$$<a^{28}> \quad <a^{44}> \quad <a^{154}>$$

$$<e>$$

1
2. (8 points total)

![Diagram]

3. (6 points total) For $\sigma = (1\ 3)(5\ 4\ 6)(4\ 2\ 6)(5\ 2\ 3)(7\ 9\ 2)$ we have:
   a) $\sigma = (1\ 3\ 4\ 2\ 7\ 9)(5)(6)$ (1 point); b) $\sigma = (7\ 9)(2\ 9)(4\ 9)(3\ 9)(1\ 9)$ (1 point); and c) $\sigma$ has order $lcm(6,1) = 6$ (1 point).

   For $\sigma = \left( \begin{array}{cccccc}
   1 & 2 & 3 & 4 & 5 & 6 \\
   4 & 6 & 1 & 9 & 8 & 2 \\
   \end{array} \right)$ we have: a) $\sigma = (1\ 4\ 9\ 3)(2\ 6)(5\ 8\ 7)$ (1 point); b) $\sigma = \left( (9\ 3)(4\ 3)(1\ 3) \right)(2\ 6)\left( (8\ 7)(5\ 7) \right)$ (1 point); and c) $\sigma$ has order $lcm(4,2,3) = 12$ (1 point).

4. (12 points total)

   a) By definition of $F$ and $\tau K$ the set $\tau K$ consists of the images of $K$ under $F$. Therefore $F$ is onto (1 point). Suppose that $\sigma, \sigma' \in K$ and $F(\sigma) = F(\sigma')$. Then $\tau \sigma = \tau \sigma'$. By (left) cancelation $\sigma = \sigma'$. Therefore $F$ is one-one. (2 points)

   b) We first show that $K \cap \tau K = \emptyset$. Since $\tau$ is odd and $K$ consists of even permutations, the set of products $\tau K$ consists of odd permutations. Since no permutation is both even and odd, $K \cap \tau K = \emptyset$ (1 point).

   We next show that $H = K \cup \tau K$. Since $K \subseteq H$, $\tau \in H$, and $H$ is closed under the operation, we conclude that $\tau K \subseteq H$. Therefore $K \cup \tau K \subseteq H$ (1 point). Conversely, suppose that $\sigma \in H$. Then $\sigma$ is even or $\sigma$ is odd.

   Case 1: $\sigma$ is even. Then $\sigma \in A_n$ which means that $\sigma \in A_n \cap H = K$.  

   

2
Case 2: $\sigma$ is odd. Since $\tau$ is odd, $\tau^{-1}$ is odd and therefore $\tau^{-1}\sigma$ is even. Since $\tau, \sigma \in H$, and $H$ is a subgroup of $G$, $\tau^{-1}\sigma \in H$. Thus $\tau^{-1}\sigma \in K$ by Case 1. This means
\[
\sigma = \text{Id}\sigma = (\tau\tau^{-1})\sigma = \tau(\tau^{-1}\sigma) \in \tau K.
\]
We have shown that for all $\sigma \in H$ either $\sigma \in K$ or $\sigma \in \tau K$. Therefore $H \subseteq \sigma \in K \cup \tau K$ (1 point). Combining the two inclusions $H = K \cup \tau K$.

c) First of all $|H| = |K| + |\tau K|$ by part b) (1 point). Since $F : K \rightarrow \tau K$ is a set bijection, $|K| = |\tau K|$ by part a) (1 point). Combining these two statements we have $|H| = |K| + |\tau K| = |K| + |K| = 2|K|$ (1 point).

d) Suppose that $H$ is subgroup of odd order. If $H$ contains an odd permutation then $|H|$ is even by part c). Since $|H|$ is odd $H$ can not contain an odd permutation. (3 points).