1. (20 points total) Let $i \in I$. Since $H_i$ is a subgroup of $G$ it follows that $e \in H_i$. Therefore $e \in \cap_{i \in I} H_i = K$ which implies that $K \neq \emptyset$ (5 points). Now let $a, b \in K$. Then $a, b \in H_i$ for all $i \in I$. Since the $H_i$’s are subgroups of $G$, by the 1-Step Subgroup Test $a^{-1}b \in H_i$ for all $i \in I$. Therefore $a^{-1}b \in \cap_{i \in I} H_i = K$ which concludes the proof that $K$ is a subgroup of $G$ by the 1-Step Subgroup Test (15 points). If the 2-Step Subgroup Test is used then closure, inverses 15 points also.

Comments: Induction does not work since the family of subgroups could very well be infinite.

2. (30 points total)
   a) Since $45 = 3^2 \cdot 5$ has six divisors, which incidently are $1, 3, 5, 9, 15, 45$, the group $G$ has six subgroups (4 points).
   b) 

<table>
<thead>
<tr>
<th>Subgroup</th>
<th>Order</th>
<th>A generator</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt;a&gt;$</td>
<td>45</td>
<td>$a^1$</td>
</tr>
<tr>
<td>$&lt;a^3&gt;$</td>
<td>15</td>
<td>$a^3$</td>
</tr>
<tr>
<td>$&lt;a^5&gt;$</td>
<td>9</td>
<td>$a^5$</td>
</tr>
<tr>
<td>$&lt;a^9&gt;$</td>
<td>5</td>
<td>$a^9$</td>
</tr>
<tr>
<td>$&lt;a^{15}&gt;$</td>
<td>3</td>
<td>$a^{15}$</td>
</tr>
<tr>
<td>$&lt;a^{45}&gt;$</td>
<td>1</td>
<td>$a^0 = e$</td>
</tr>
</tbody>
</table>

   (8 points).

c) Since the integers $1 \leq k \leq 15$ which are relatively prime to 15 are the eight integers $1, 2, 4, 7, 8, 11, 13, 14$, the generators of the subgroup $<a^3>$ of $G$ of order 15 are $(a^3)^k$ for these values of $k$. Thus the answer is:

$$a^3, a^6, a^{12}, a^{21}, a^{24}, a^{33}, a^{39}, a^{42} \quad (6 \text{ points})$$
d) Since \( \gcd(45, 250) = \gcd(3^2 \cdot 5, 2 \cdot 5^3) = 5 \) it follows that \( d = 5 \) (3 points). Thus \( <a^{250}> = <a^5> \) and has the nine elements

\[
    e = a^0, a^5, a^{10}, a^{15}, a^{20}, a^{25}, a^{30}, a^{35}, a^{40} \quad (3 \text{ points}).
\]

e)

\[
\begin{align*}
    &<a^1> \\
    & \downarrow \\
    &<a^3> \quad <a^5> \\
    & | \quad | \\
    &<a^9> \quad <a^{15}> \\
    & \downarrow \quad \downarrow \\
    &<a^0> \\
\end{align*}
\]

(6 points).

3. (25 points total) \( \text{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^0 & 0 \\ 0 & 3^0 \end{pmatrix} \in H \), here \( m = n = b = 0 \), so \( H \neq \emptyset \) (5 points). For \( a, b, d \in \mathbb{R} \), where \( a, d \neq 0 \), we have

\[
\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & -ba^{-1}d^{-1} \\ 0 & d^{-1} \end{pmatrix}.
\]

(8 points for an inverse calculation). In particular \( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G \). Thus \( H \subseteq G \). Since

\[
\begin{pmatrix} 2^m & b \\ 0 & 3^n \end{pmatrix}^{-1} \begin{pmatrix} 2^{m'} & b' \\ 0 & 3^{n'} \end{pmatrix} = \begin{pmatrix} 2^{-m+m'} & b'' \\ 0 & 3^{-(n+n')} \end{pmatrix}^{-1} \in H,
\]

where \( b'' = 2^{-m}b' - b2^{-m}3^{-n+n'} \), we conclude that \( H \) is a subgroup of \( G \) by the 1-step subgroup test (12 points). If the 2-Step Subgroup Test is used then closure, inverses 12 points also.
4. (25 points total) $f = (1\ 3\ 5\ 6)(2\ 4\ 6\ 9\ 7)(7\ 8\ 9)$ so a) $f = (1\ 3\ 5\ 4)(2\ 6\ 9)(7\ 8)$ (10 points). b) Using part a) we write $f = (5\ 4)(3\ 4)(1\ 4)(6\ 9)(2\ 9)(7\ 8)$ (5 points). $f$ is even since it can be written as a product of an even number (6) of 2-cycles (5 points). d) Using part a) we have the disjoint cycle decomposition $f^2 = (1\ 5)(3\ 4)(2\ 9\ 6)$ (5 points).