1. (20 points)

(a) $P$ is the set (subspace) of solutions to the system of linear equations $3x + 2y - z + 5w = 0$ and consists of all vectors in $\mathbb{R}^4$ perpendicular to $u$. (10)

(b) Row reduction yields $x + \frac{2}{3}y - \frac{1}{3}z + \frac{5}{3}w = 0$ and thus

\[
x = -\frac{2}{3}y + \frac{1}{3}z - \frac{5}{3}w \\
y = 1y + 0z + 0w \\
z = 0y + 1z + 0w \\
w = 0y + 0z + 1w
\]

which in vector form is

\[
\begin{pmatrix}
x \\ y \\ z \\ w
\end{pmatrix} = y \begin{pmatrix}
-\frac{2}{3} \\
1 \\
0 \\
0
\end{pmatrix} + z \begin{pmatrix}
\frac{1}{3} \\
0 \\
1 \\
0
\end{pmatrix} + w \begin{pmatrix}
-\frac{5}{3} \\
0 \\
0 \\
1
\end{pmatrix}.
\]

Thus a basis for $P$ is

\[
\left\{ \begin{pmatrix}
-\frac{2}{3} \\
1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
\frac{1}{3} \\
0 \\
1 \\
0
\end{pmatrix}, \begin{pmatrix}
-\frac{5}{3} \\
0 \\
0 \\
1
\end{pmatrix} \right\},
\]

(10) or clearing fractions

\[
\left\{ \begin{pmatrix}
-2 \\
3 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
1 \\
0 \\
3 \\
0
\end{pmatrix}, \begin{pmatrix}
-5 \\
0 \\
0 \\
3
\end{pmatrix} \right\}.
\]

Comment: There are many possible answers.

2. (20 points) First some very basic observations about matrix multiplication. Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$. Suppose that $A$ is an $m \times n$ matrix (with real coefficients). Then

\[
Ae_j \quad \text{is the } j^{th} \text{ column of } A.
\]

Suppose that $A$ is an $n \times m$ matrix. Then

\[
e_i^t A \quad \text{is the } i^{th} \text{ row of } A.
\]
Now suppose that \( m = n \), write \( A = (a_{ij}) \), and consider the bilinear form on \( \mathbb{R}^n \) defined by
\[
\langle u, v \rangle = u^t A v
\]
for all \( u, v \in \mathbb{R}^n \). Then the calculation
\[
\langle e_i, e_j \rangle = e_i^t A e_j = e_i^t (A e_j) = e_i^t \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} = a_{ij},
\]
where the latter is identified with the 1\(\times\)1 matrix with entry \( a_{ij} \), shows that
\[
\langle e_i, e_j \rangle = a_{ij}
\]
for all \( 1 \leq i, j \leq n \).

Suppose that \( \langle , \rangle \) is symmetric. Then \( a_{ij} = \langle e_i, e_j \rangle = \langle e_j, e_i \rangle = a_{ji} \) for all \( 1 \leq i, j \leq n \) shows that \( A \) is symmetric.

**Comment:** Some solutions ended "\( u^t A v = u^t A^t v \) for all \( u, v \in \mathbb{R}^n \), and therefore \( A = A^t \)." There is a significant gap in this proof.

3. **(20 points)** Suppose that \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) defines an inner product on \( \mathbb{R}^2 \) by (1). Then
\( A \) is symmetric by Exercise 4.2.7. Thus \( c = b \). Using the solution to Exercise 4.2.7 we observe that
\[
0 < \langle e_1, e_1 \rangle = a_{11} = a \quad \text{and} \quad 0 < \langle e_2, e_2 \rangle = a_{22} = d.
\]
Since \( b = c \) and \( a, d \neq 0 \) (as they are positive), it follows that
\[
\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \rangle = ax^2 + 2bxy + dy^2 = a(x + \frac{b}{a}y)^2 + (d - \frac{b^2}{a})y^2
\]
for all \( \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \). By virtue of (2) we have
\[
0 < \langle \begin{pmatrix} -b \\ a \end{pmatrix}, \begin{pmatrix} -b \\ a \end{pmatrix} \rangle = (d - \frac{b^2}{a})a^2
\]
from which we deduce \( d - \frac{b^2}{a} > 0 \), or equivalently \( ad - b^2 > 0 \), since \( a^2 > 0 \). (10)

Conversely, suppose that \( A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \) where \( a, d, ad - b^2 > 0 \). Since \( A \) is symmetric (1) defines a symmetric bilinear form on \( \mathbb{R}^2 \). Let \( \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \). Then (2) holds, and the right hand expression is a non-negative real number since it is the sum of products of
non-negative real numbers. Suppose that the right hand expression is 0. Since the two summands are non-negative, it follows that
\[ a(x + \frac{b}{a}y)^2 = (d - \frac{b^2}{a})y^2 = 0, \]
and thus
\[ (x + \frac{b}{a}y)^2 = y^2 = 0 \]
as \( a, d - \frac{b^2}{a} \neq 0 \). Therefore \( y = 0 \), and hence \( x^2 = 0 \). We have shown that \( \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0} \). (10)

4. (20 points) Suppose that \( A \) is a \( 2 \times 2 \) matrix with real coefficients as in Exercise 4.2.8. Then \( A \) determines an inner product on \( \mathbb{R}^2 \) by (1). Such a matrix is \( A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \) where \( a, d \) are positive real numbers. Observe that
\[ \langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rangle = ax_1y_1 + dx_2y_2 \]
for all \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2 \).

(a) \( ||u|| = \sqrt{a + d} \). Thus take \( a = 2000, d = 1 \) and \( a = 1, d = 2000 \). In either case \( ||u|| = \sqrt{2001} \). These choices give different inner products; indeed in the first case \( || \begin{pmatrix} 1 \\ 0 \end{pmatrix} || = \sqrt{2000} \) and in the second \( || \begin{pmatrix} 1 \\ 0 \end{pmatrix} || = 1 \). (6)

(b) and (c). \( \langle u, v \rangle = a - d \) and \( ||u|| = \sqrt{a + d} = ||v|| \). Thus
\[ \cos \theta = \frac{\langle u, v \rangle}{||u|| ||v||} = \frac{a - d}{a + d}. \]
For part (a) we need to solve \( \frac{a - d}{a + d} = \frac{1}{2} \), or equivalently \( a = 3d \). Take \( d = 1, a = 3 \) for example. (7) For part (b) we need to solve \( \frac{a - d}{a + d} = \frac{\sqrt{3}}{2} \), or equivalently \( a(2 - \sqrt{3}) = d(2 + \sqrt{3}) \). Take \( d = 1, a = \frac{2 + \sqrt{3}}{2 - \sqrt{3}} \) for example. (7)

Comment: Some students assumed that \( ||u|| = \sqrt{2} = ||v|| \) in solving parts (b) and (c). This is the case for the standard inner product. These lengths depend on the choice of \( a \) and \( d \).
5. (20 points) Let $u, v \in V$. From the calculation

$$
||u + v||^2 = \langle u + v, u + v \rangle \\
= \langle u + v, u \rangle + \langle u + v, v \rangle \\
= (\langle u, u \rangle + \langle v, u \rangle) + (\langle u, v \rangle + \langle v, v \rangle) \\
= ||u||^2 + 2 \langle u, v \rangle + ||v||^2
$$

we conclude that

$$
||u + v||^2 = ||u||^2 + 2 \langle u, v \rangle + ||v||^2.
$$

(3)

"If". Suppose that $\langle u, v \rangle = 0$. Then $2 \langle u, v \rangle = 0$ and thus $||u + v||^2 = ||u||^2 + ||v||^2$ by (3). (10)

"Only if". Suppose that $||u + v||^2 = ||u||^2 + ||v||^2$. Then $2 \langle u, v \rangle = 0$ by (3) and hence $\langle u, v \rangle = 0$. (10)