MATH 425 **Written Homework 2 Solution** **Radford** **02/07/08**

1. (**20 points**) We have noted that orthogonal complements are subspaces. Thus $S^\perp$ is a subspace of $\mathbf{R}^4$.

(a) Let $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbf{R}^4$. Then $\mathbf{v} \in S^\perp$ if and only if $<\mathbf{u}_1, \mathbf{v}> = 0 = <\mathbf{u}_2, \mathbf{v}>$ by Lemma 4.3.1. Thus $\mathbf{v} \in S^\perp$ if and only if

$$\begin{aligned} 1x + 2y + 1z + 0w &= 0 \\ 4x + 1y + 2z + 3w &= 0. \end{aligned}$$

Row reduction yields

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 4 & 1 & 2 & 3 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 1 & 0 & 3/7 & 6/7 \\ 0 & 1 & 2/7 & -3/7 \end{pmatrix}.$$

Therefore $\mathbf{v} \in S^\perp$ if and only if

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = z \begin{pmatrix} -3/7 \\ -2/7 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -6/7 \\ 3/7 \\ 0 \\ 1 \end{pmatrix}$$

which means $\{\begin{pmatrix} -3/7 \\ -2/7 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -6/7 \\ 3/7 \\ 0 \\ 1 \end{pmatrix}\} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $S^\perp$. (**10**)

(b) $\mathbf{q}_1 = \dfrac{7\mathbf{v}_1}{||7\mathbf{v}_1||} = \dfrac{1}{\sqrt{62}} \begin{pmatrix} -3 \\ -2 \\ 7 \\ 0 \end{pmatrix}$.

$$\begin{aligned} \mathbf{w_2} &= \mathbf{v}_2 - <\mathbf{v}_2, \mathbf{q}_1>\mathbf{q}_1 \\ &= \begin{pmatrix} -6/7 \\ 3/7 \\ 0 \\ 1 \end{pmatrix} - <\begin{pmatrix} -6/7 \\ 3/7 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{62}} \begin{pmatrix} -3 \\ -2 \\ 7 \\ 0 \end{pmatrix}> \frac{1}{\sqrt{62}} \begin{pmatrix} -3 \\ -2 \\ 7 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -6/7 \\ 3/7 \\ 0 \\ 1 \end{pmatrix} - \frac{12}{62 \cdot 7} \begin{pmatrix} -3 \\ -2 \\ 7 \\ 0 \end{pmatrix} \end{aligned}$$

$$= \frac{1}{31{\cdot}7}\begin{pmatrix} -6{\cdot}31 \\ 3{\cdot}31 \\ 0 \\ 7{\cdot}31 \end{pmatrix} - \frac{6}{31{\cdot}7}\begin{pmatrix} -3 \\ -2 \\ 7 \\ 0 \end{pmatrix}$$

$$= \frac{1}{31{\cdot}7}\begin{pmatrix} -168 \\ 105 \\ -6{\cdot}7 \\ 31{\cdot}7 \end{pmatrix}$$

$$= \frac{1}{31}\begin{pmatrix} -24 \\ 15 \\ -6 \\ 31 \end{pmatrix}.$$

Thus $\mathbf{q}_2 = \dfrac{31\mathbf{w}_2}{||31\mathbf{w}_2||} = \dfrac{1}{\sqrt{1798}}\begin{pmatrix} -24 \\ 15 \\ -6 \\ 31 \end{pmatrix}$. An answer is $\{\mathbf{q}_1, \mathbf{q}_2\}$. (**10**)

2. (**20 points**) From the calculation

$$<\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix}> = \begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} 1 & 2 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} z+2w \\ 2z+7w \end{pmatrix} = xz{+}2xw{+}2yz{+}7yw$$

we have

$$<\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix}> = xz + 2xw + 2yz + 7yw \qquad (1)$$

for all $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix} \in \mathbf{R}^2$. We apply the Gram-Schmidt process to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for $\mathbf{R}^2$.

By (1) note that $<\mathbf{e}_1, \mathbf{e}_1> = <\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}> = 1$. Therefore $\mathbf{q}_1 = \mathbf{e}_1$. (**10**) By (1) again

$$\mathbf{w}_2 = \mathbf{e}_2 - <\mathbf{e}_2, \mathbf{q}_1>\mathbf{q}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - <\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}>\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Using (1) again we see $<\mathbf{w}_2, \mathbf{w}_2> = <\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}> = 3$ and thus $\mathbf{q}_2 = \dfrac{1}{\sqrt{3}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.
One answer is $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \dfrac{1}{\sqrt{3}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}\}$. (**10**)

3. (**20 points**) Since $V = S \oplus T$ is an orthogonal sum, by definition any $\mathbf{v} \in V$ can be written $\mathbf{v} = \mathbf{s} + \mathbf{t}$ for some $\mathbf{s} \in S$, $\mathbf{t} \in T$ and for all $\mathbf{s} \in S$ and $\mathbf{t} \in T$ it follows that $<\mathbf{s}, \mathbf{t}> = 0$.

Let $\mathbf{t} \in T$. Thus $<\mathbf{t}, \mathbf{s}> = <\mathbf{s}, \mathbf{t}> = 0$ for all $\mathbf{s} \in S$ since inner products are symmetric and by definition of orthogonal sum. Therefore $\mathbf{t} \in S^{\perp}$. We have shown that $T \subseteq S^{\perp}$. (**10**)

Suppose that $\mathbf{v} \in S^{\perp}$. Then $\mathbf{v} = \mathbf{s} + \mathbf{t}$ for some $\mathbf{s} \in S$ and $\mathbf{t} \in T$. Since $\mathbf{v}, \mathbf{t} \in S^{\perp}$ and $\mathbf{s} \in S$ we have

$$0 = <\mathbf{v}, \mathbf{s}> = <\mathbf{s} + \mathbf{t}, \mathbf{s}> = <\mathbf{s}, \mathbf{s}> + <\mathbf{t}, \mathbf{s}> = <\mathbf{s}, \mathbf{s}> + 0 = <\mathbf{s}, \mathbf{s}>$$

which means that $\mathbf{s} = 0$. Therefore $\mathbf{v} = \mathbf{t} \in T$. We have shown that $S^{\perp} \subseteq T$, and thus $T = S^{\perp}$. (**10**)

4. (**20 points**) We first show that $(\mathbf{v} - \mathbf{s}) \perp S$. This is equivalent to showing that $<\mathbf{v} - \mathbf{s}, \mathbf{q}_i> = 0$ for all $1 \leq i \leq r$ by Lemma 4.3.1. The preceding equation holds since

$$\begin{aligned} <\mathbf{v} - \mathbf{s}, \mathbf{q}_i> &= <\mathbf{v} - <\mathbf{v}, \mathbf{q}_1>\mathbf{q}_1 - \cdots - <\mathbf{v}, \mathbf{q}_r>\mathbf{q}_r, \mathbf{q}_i> \\ &= <\mathbf{v}, \mathbf{q}_i> - <\mathbf{v}, \mathbf{q}_1><\mathbf{q}_1, \mathbf{q}_i> - \cdots - <\mathbf{v}, \mathbf{q}_r><\mathbf{q}_r, \mathbf{q}_i> \\ &= <\mathbf{v}, \mathbf{q}_i> - <\mathbf{v}, \mathbf{q}_i><\mathbf{q}_i, \mathbf{q}_i> \\ &= <\mathbf{v}, \mathbf{q}_i> - <\mathbf{v}, \mathbf{q}_i> \\ &= 0 \end{aligned}$$

for all $1 \leq i \leq r$. (**10**)

Let $\mathbf{s}' \in S$. Then $\mathbf{s} - \mathbf{s}' \in S$, since S is a subspace of V. We have just shown that $<\mathbf{v} - \mathbf{s}, \mathbf{s} - \mathbf{s}'> = 0$. Thus by Exercise 5 of Written Homework 1 we compute

$$||\mathbf{v} - \mathbf{s}'||^2 = ||(\mathbf{v} - \mathbf{s}) + (\mathbf{s} - \mathbf{s}')||^2 = ||\mathbf{v} - \mathbf{s}||^2 + ||\mathbf{s} - \mathbf{s}'||^2 \geq ||\mathbf{v} - \mathbf{s}||^2.$$

Since $||\mathbf{v} - \mathbf{s}'||^2 \geq ||\mathbf{v} - \mathbf{s}||^2$ it follows that $||\mathbf{v} - \mathbf{s}'|| \geq ||(\mathbf{v} - \mathbf{s})||$ as lengths are non-negative numbers. Our calculation shows that if $\mathbf{s}'$ is also a vector in S closest to $\mathbf{v}$, in which case $||\mathbf{v} - \mathbf{s}'|| = ||(\mathbf{v} - \mathbf{s})||$, then $||\mathbf{s} - \mathbf{s}'||^2 = 0$ and consequently $\mathbf{s} = \mathbf{s}'$. (**10**)

5. (**20 points**) This exercise is application of formulas. $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 4 \\ 3 \end{pmatrix}$.

(a) Thus $<\mathbf{x}, \mathbf{y}> = 10$, $\overline{x} = \frac{1}{4}$, $\overline{y} = 2$, and $\sigma^2 = (-2 - \frac{1}{4})^2 + (0 - \frac{1}{4})^2 + (1 - \frac{1}{4})^2 + (2 - \frac{1}{4})^2 = \frac{35}{4}$. Consequently

$$m = \frac{<\mathbf{x}, \mathbf{y}> - 4\overline{x}\,\overline{y}}{\sigma^2} = \frac{10 - 4 \cdot \frac{1}{4} \cdot 2}{\frac{35}{4}} = \frac{32}{35} \quad \text{and} \quad b = \overline{y} - m\overline{x} = 2 - \frac{32}{35} \cdot \frac{1}{4} = \frac{62}{35}.$$

Thus $y = \frac{32}{35}x + \frac{62}{35}$. (**10**)

(b) $A = \begin{pmatrix} 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ and thus $A^tA = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & 0 & 1 & 2 \\ 4 & 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 9 \\ 1 & 9 & 1 \\ 9 & 1 & 33 \end{pmatrix}$.

By Theorem 4.6.1 the polynomial $f(x) = a_0 + a_1x + a_2x^2$ is determined by the linear system $A^tA \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = A^t\mathbf{y}$; that is

$$\begin{pmatrix} 4 & 1 & 9 \\ 1 & 9 & 1 \\ 9 & 1 & 33 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & 0 & 1 & 2 \\ 4 & 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \\ 16 \end{pmatrix}.$$

There are various ways of solving this system; using row reduction or by finding the inverse of A^tA. The latter can be done easily enough by computing the classical adjoint. In any case $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{106}{55} \\ \frac{199}{220} \\ -\frac{3}{44} \end{pmatrix}$. Thus $f(x) = \frac{106}{55} + \frac{199}{220}x - \frac{3}{44}x^2$. (**10**)