MATH 425 Written Homework 7 Solution Radford 04/21/08

1. (**20 points**) $A = \begin{pmatrix} 9 & 0 & 0 & 0 \\ 9 & 8 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 12 & 11 & 10 & 9 \end{pmatrix}$. Therefore $c_A(x) = (9-x)^3(8-x)$ from which we deduce $\mathrm{Dim}\,\mathcal{N}((A-9\mathrm{I}_4)^3) = 3$, $\mathrm{Dim}\,\mathcal{N}(A-8\mathrm{I}_4) = 1$. Note that $v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -11 \end{pmatrix}$ forms a basis for the space of eigenvectors for A belonging to $\lambda - 8$, hence forms a basis for $\mathcal{N}(A-8\mathrm{I}_4)$.

Now $\mathcal{N}((A-9\mathrm{I}_4)^3) = \mathcal{R}(A-8\mathrm{I}_4)$ has basis $\{e_4, e_3, e_1+9e_2\}$. Under multiplication by $A - 9\mathrm{I}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 9 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 12 & 11 & 10 & 0 \end{pmatrix}$ observe that $e_4 \longrightarrow 0$, $e_3 \longrightarrow 10e_4 \longrightarrow 0$, and $e_1 + 9e_2 \longrightarrow 111e_4 \longrightarrow 0$. Therefore $v_2 = e_1 + 9e_2 - \frac{111}{10}e_3, v_3 = 10e_4$ form an independent set of eigenvectors for A belonging to $\lambda = 9$. Set $v_4 = e_3$. Let Let $S = (v_1 \cdots v_4) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 9 & 0 & 0 \\ 0 & -\dfrac{111}{10} & 0 & 1 \\ -11 & 0 & 10 & 0 \end{pmatrix}$ (**12**) and $J = \begin{pmatrix} 8 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 9 & 1 \\ 9 & 0 & 0 & 9 \end{pmatrix}$. (**8**)

Comment: I was not particular about the order of the blocks.

2. (**20 points**) $A = \begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 9 & 0 & 0 & 9 \end{pmatrix}$. Therefore $c_A(x) = (9-x)^2x^2$ from which we deduce $\mathrm{Dim}\,\mathcal{N}((A-9\mathrm{I}_4)^2) = 2 = \mathrm{Dim}\,\mathcal{N}(A^2)$. By inspection $\{e_2, e_3\}$ form a basis for the space of eigenvectors for A belonging to $\lambda = 0$, hence form a basis for $\mathcal{N}(A^2)$. Let $v_1 = e_2, v_2 = e_3$. Now $\mathcal{R}(A^2) = \mathcal{N}((A-9\mathrm{I}_4)^2)$ has basis $\{9e_1+9e_4, e_4\}$. Let $v_4 = 9e_1 + 9e_4$ and $v_3 = (A-9\mathrm{I}_4)v_4 = 81e_4$. Then $\{v_3, v_4\}$ is a basis for $\mathcal{N}((A-\mathrm{I}_3)^2)$. Let $S = (v_1 \cdots v_4) = \begin{pmatrix} 0 & 0 & 0 & 9 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 81 & 9 \end{pmatrix}$ (**12**) and $J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 1 \\ 9 & 0 & 0 & 9 \end{pmatrix}$. (**8**)

Comment: Note that $\{e_1, e_4\}$ is a basis for $\mathcal{N}((A-9\mathrm{I}_4)^2)$ as well. Thus $v_4 = e_1$ and $v_3 = 9e_4$ work also. I was not particular about the order of the blocks.

3. (**20 points**) We are to find $c_A(x)$ and $m_A(x)$, where A is the matrix of the reflection $R : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ through $S = \text{span}(\begin{pmatrix} 3 \\ 7 \end{pmatrix})$.

Now R is a reflection through a line in $\mathbf{R}^2$. Therefore A has eigenvalues $1, -1$. Thus $(x-1)(x+1)$ divides $c_A(x)$, since the roots of $c_A(x)$ are the eigenvalues of A, and $(x-1)(x+1)$ divides $m_A(x)$ by part d) of Proposition 7.6.1. (**10**) Since $m_A(x)$ divides $c_A(x)$ by part c) of the same, both have highest coefficient 1, and the degree of $c_A(x)$ is 2, $c_A(x) = (x-1)(x+1) = m_A(x)$. (**10**)

Comment: The matrix of R is $A = \begin{pmatrix} -\frac{20}{29} & \frac{21}{29} \\ \frac{21}{29} & \frac{20}{29} \end{pmatrix}$. The solution can be based on this.

4. (**20 points**) Let $m \geq 0$. Then $D(1) = 0$ and $D(x^m) = mx^{m-1}$ for all $m \geq 1$. Let $\mathcal{B} = \{1, x, \ldots, x^n\}$ be the natural basis for P^n. Then $[D]_{\mathcal{B}}$ is an $(n+1)\times(n+1)$ upper triangular matric with zeros on the diagonal. Therefore $c_D(x) = c_{[D]_{\mathcal{B}}}(x) = (-x)^{n+1}$. (**8**)

To find $m_D(x)$ one can verify Exercise 8.1.5 and use Proposition 7.6.1 to show that $m_D(x) = x^{n+1}(= \pm c_D(x))$. A more direct way is to first note that $D^{n+1} = 0$ and $D^n \neq 0$. In particular $\{I, \ldots, D^{n+1}\}$ is dependent. We will show $\{I, \ldots, D^n\}$ is independent. Since $D^{n+1} = 0$ by definition $m_D(x) = x^{n+1}$.

Suppose that $\{I, \ldots, D^n\}$ is dependent. Then $a_0I + a_1D + \cdots + a_{n-1}D^n = 0$ for some $a_0, \ldots, a_n \in \mathbf{R}$ not all of which are zero. Thus $a_mD^m + \cdots + a_nD^n = 0$ where $0 \leq m \leq n$ and $a_m \neq 0$. Applying D^{n-m} to both sides of the preceding equation $a_mD^n = 0$. Since $D^n \neq 0$ necessarily $a_m = 0$, a contradiction. Thus $\{I, \ldots, D^n\}$ is independent after all. (**12**)

5. (**20 points**) Let $\mathbf{v} \in V$, $\lambda \in \mathbf{R}$ and suppose that $T(v) = \lambda v$. Then $T^m(\mathbf{v}) = \lambda^m\mathbf{v}$ for all $m \geq 0$ by induction on m. Since $T^0(\mathbf{v}) = \mathbf{v} = \lambda^0\mathbf{v}$ the assertion follows for $m = 0$. (**8**)

Suppose that $m \geq 0$ and $T^m(\mathbf{v}) = \lambda^m\mathbf{v}$. The calculation

$$T^{m+1}(\mathbf{v}) = T(T^m(\mathbf{v})) = T(\lambda^m\mathbf{v}) = \lambda^mT(\mathbf{v}) = \lambda^m(\lambda\mathbf{v}) = \lambda^{m+1}\mathbf{v}$$

shows that the assertion holds for $m+1$. Thus the assertion holds for all $m \geq 0$ by induction. (**8**)

Next let $p(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbf{R}[x]$. Then the calculation

$$\begin{aligned} p(T)(\mathbf{v}) &= (a_0I + a_1T + \cdots + a_nT^n)(\mathbf{v}) \\ &= a_0I(\mathbf{v}) + a_1T(\mathbf{v}) + \cdots + a_nT(\mathbf{v}) \\ &= a_0\mathbf{v} + a_1(\lambda\mathbf{v}) + \cdots + a_n(\lambda^n\mathbf{v}) \\ &= (a_0 + a_1\lambda + \cdots + a_n\lambda^n)\mathbf{v} \\ &= p(\lambda)\mathbf{v} \end{aligned}$$

shows that $p(T)(\mathbf{v}) = p(\lambda)\mathbf{v}$. (**8**)

Suppose that λ is an eigenvalue for T and $p(T) = 0$. Let $\mathbf{v}$ be a non-zero eigenvector for T belonging to λ. Since $\mathbf{v} \neq \mathbf{0}$, $0 = p(T)(\mathbf{v}) = p(\lambda)\mathbf{v}$ which shows that $p(\lambda) = 0$. (**4**)

Comment: We have shown that if $p(T) = 0$ the eigenvalues of T are roots of $p(x)$. A root of $p(x)$ is not necessarily an eigenvalue for T. For suppose that V is finite-dimensional. Then T has only finitely many eigenvalues. Suppose $a \in \mathbf{R}$ is not of of them. Note $p_a(x) = p(x)(x - a)$ also satisfies $p_a(T) = p(T)(T - aI) = 0$.