1. Consider the vector space $P^2 = \{a+bx+cx^2 \mid a, b, c \in \mathbb{R}\}$ of polynomials of degree at most two as an inner product space where $\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x) \, dx$ for all $f(x), g(x) \in P^2$. Let $S$ be the span of $x$.

(a) Find $\langle x^\ell, x^m \rangle$, where $\ell + m$ is odd.

(b) Find an orthonormal basis for $S^\perp$.

2. Let $V$ be any inner product space and let $S$ be a finite-dimensional subspace with orthonormal basis $\{q_1, \ldots, q_r\}$. Suppose $v \in V$ and set $u = \langle v, q_1 \rangle q_1 + \cdots + \langle v, q_r \rangle q_r$.

(a) Show that $(v - u) \perp S$.

(b) Show that $u$ is a closest vector in $S$ to $v$.

You may use: If $u, v \in V$ and $\langle u, v \rangle = 0$ then $\|u + v\|^2 = \|u\|^2 + \|v\|^2$.

3. Consider $\mathbb{R}^3$ as an inner product space with the standard inner product and let $S$ be the subspace with basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$.

(a) Find the vector in $S$ closest to $b = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

(b) Find a vector $x = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ which is a least squares solution to $Ax = b$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 2 & -1 \end{pmatrix}$.

(c) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the orthogonal projection of $\mathbb{R}^3$ onto $S$. Find a $3 \times 3$ matrix $A$ such that $T(v) = Av$ for all $v \in \mathbb{R}^3$. 

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4. Find the matrix of the rotation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which satisfies $T \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$.

5. Find the spectral decomposition of $A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$, given $c_A(x) = -(x + 2)^2(x - 4)$.

6. The only complex eigenvalue which $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ -1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 \end{pmatrix}$ has is $\lambda = 0$. Find an invertible matrix $S$ and Jordan matrix $J$ such that $A = SJS^{-1}$.

7. $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. Find the characteristic polynomial of $A$, an invertible matrix $S$, and Jordan matrix $J$ such that $A = SJS^{-1}$.

8. Suppose $v \in V$, $n > 0$ and $T^n(v) = 0 \neq T^{n-1}(v)$.

   (a) Show that $\{v, T(v), \ldots, T^{n-1}(v)\}$ is linearly independent.

   (b) Show that $\{I, T, \ldots, T^{n-1}\}$ is linearly independent.

   (c) Suppose that $T^n = 0$. Show that $m_T(x) = x^n$. 