MATH 425 FINAL EXAMINATION 05/06/08 8AM–10AM TH 117

Name (print) ______________________________

(1) *Return* this exam copy with your exam booklet. (2) *Write* your solutions in your exam booklet. (3) *Show* your work. (4) There are *eight questions* on this exam. (5) Each question counts 25 points. (6) You are expected *to abide by* the University's rules concerning academic honesty.

Unless otherwise stated, V is a vector space over $\mathbf{R}$ and $T : V \longrightarrow V$ is linear.

---

1. Consider the vector space $\mathbf{P}^2 = \{a+bx+cx^2 \mid a, b, c \in \mathbf{R}\}$ of polynomials of degree at most two as an inner product space where $<f(x), g(x)> = \int_{-1}^{1} f(x)g(x)\,dx$ for all $f(x), g(x) \in \mathbf{P}^2$. Let S be the span of x.

(a) Find $<x^\ell, x^m>$, where $\ell + m$ is odd.

(b) Find an orthonormal basis for $S^\perp$.

2. Let V be *any* inner product space and let S be a finite-dimensional subspace with orthonormal basis $\{\mathbf{q}_1, \ldots, \mathbf{q}_r\}$. Suppose $\mathbf{v} \in V$ and set $\mathbf{u} = <\mathbf{v}, \mathbf{q}_1>\mathbf{q}_1 + \cdots + <\mathbf{v}, \mathbf{q}_r>\mathbf{q}_r$.

(a) Show that $(\mathbf{v} - \mathbf{u}) \perp S$.

(b) Show that $\mathbf{u}$ is a closest vector in S to $\mathbf{v}$.

You may use: If $\mathbf{u}, \mathbf{v} \in V$ and $<\mathbf{u}, \mathbf{v}> = 0$ then $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.

3. Consider $\mathbf{R}^3$ as an inner product space with the standard inner product and let S be the subspace with basis $\{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\}$.

(a) Find the vector in S closest to $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

(b) Find a vector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2$ which is a least squares solution to $A\mathbf{x} = \mathbf{b}$, where
$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 2 & -1 \end{pmatrix}.$$

(c) Let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the orthogonal projection of $\mathbf{R}^3$ onto S. Find a $3{\times}3$ matrix A such that $T(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbf{R}^3$.

4. Find the matrix of the rotation $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ which satisfies $T\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$.

5. Find the spectral decomposition of $A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$, given $c_A(x) = -(x+2)^2(x-4)$.

6. The only complex eigenvalue which $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ -1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 \end{pmatrix}$ has is $\lambda = 0$. Find an invertible matrix S and Jordan matrix J such that $A = SJS^{-1}$.

7. $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. Find the characteristic polynomial of A, an invertible matrix S, and Jordan matrix J such that $A = SJS^{-1}$.

8. Suppose $\mathbf{v} \in V$, $n > 0$ and $T^n(\mathbf{v}) = \mathbf{0} \neq T^{n-1}(\mathbf{v})$.

(a) Show that $\{\mathbf{v}, T(\mathbf{v}), \ldots, T^{n-1}(\mathbf{v})\}$ is linearly independent.

(b) Show that $\{I, T, \ldots, T^{n-1}\}$ is linearly independent.

(c) Suppose that $T^n = 0$. Show that $m_T(x) = x^n$.