**MATH 425 FINAL EXAMINATION SOLUTION 05/12/08**

Name (print) ____________________________________

(1) *Return* this exam copy with your exam booklet. (2) *Write* your solutions in your exam booklet. (3) *Show* your work. (4) There are *eight questions* on this exam. (5) Each question counts 25 points. (6) You are expected *to abide by* the University's rules concerning academic honesty.

*Unless otherwise stated, $V$ is a vector space over* $\mathbf{R}$ *and* $T : V \longrightarrow V$ *is linear.*

---

1. Consider the vector space $\mathbf{P}^2 = \{a+bx+cx^2 \mid a, b, c \in \mathbf{R}\}$ of polynomials of degree at most two as an inner product space where $<f(x), g(x)> = \int_{-1}^{1} f(x)g(x)\,dx$ for all $f(x), g(x) \in \mathbf{P}^2$. Let $S$ be the span of $x$.

(a) Find $<x^\ell, x^m>$, where $\ell + m$ is odd.

*Solution*: Since $\ell + m$ is odd, $\ell + m + 1$ is even. Thus

$$<x^\ell, x^m> = \int_{-1}^{1} x^\ell x^m\,dx = \left.\frac{x^{\ell+m+1}}{\ell+m+1}\right|_{-1}^{1} = \frac{1}{\ell+m+1}(1-1) = 0. \ (\mathbf{8\ points}).$$

(b) Find an orthonormal basis for $S^\perp$.

*Solution*: By part (a) $1, x^2 \in S^\perp$. Since $\mathrm{Dim}\, S + \mathrm{Dim}\, S^\perp = 3$, $\mathrm{Dim}\, S^\perp = 2$ and therefore $\{1, x^2\}$ is a basis for $S^\perp$. (**4 points**). One can derive this basis for $S^\perp$ from part (a) by the observation $0 = <a + bx + cx^2, x> = a<1, x> + b<x, x> + c<x^2, x> = b<x, x>$ if and only if $b = 0$.

We apply the Gram-Schmidt process to this basis to obtain the orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2\} = \{\frac{1}{\sqrt{2}}, \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)\}$ for $S^\perp$; $\mathbf{q}_1$ (**4 points**), $\mathbf{w}_2$ (**4 points**), $\mathbf{q}_2$ (**5 points**).

2. Let $V$ be *any* inner product space and let $S$ be a finite-dimensional subspace with orthonormal basis $\{\mathbf{q}_1, \ldots, \mathbf{q}_r\}$. Suppose $\mathbf{v} \in V$ and set $\mathbf{u} = <\mathbf{v}, \mathbf{q}_1>\mathbf{q}_1 + \cdots + <\mathbf{v}, \mathbf{q}_r>\mathbf{q}_r$.

(a) Show that $(\mathbf{v} - \mathbf{u}) \perp S$.

*Solution*: Let $T = \mathbf{R}(\mathbf{v} - \mathbf{u})$ be the one-dimensional subspace of $V$ spanned by $\mathbf{v} - \mathbf{u}$. Since $T^\perp$ is a subspace of $V$, to show that $(\mathbf{v} - \mathbf{u}) \perp S$, that is $S \subseteq T^\perp$, we need only show that $(\mathbf{v} - \mathbf{u}) \perp \mathbf{q}_i$ for all $1 \le i \le r$. The calculation

$$\begin{aligned}
&<\mathbf{v} - (<\mathbf{v}, \mathbf{q}_1>\mathbf{q}_1 + \cdots + <\mathbf{v}, \mathbf{q}_r>\mathbf{q}_r>), \mathbf{q}_i> \\
&= <\mathbf{v}, \mathbf{q}_i> - <\mathbf{v}, \mathbf{q}_1><\mathbf{q}_1, \mathbf{q}_i> - \cdots - <\mathbf{v}, \mathbf{q}_r><\mathbf{q}_r, \mathbf{q}_i> \\
&= <\mathbf{v}, \mathbf{q}_i> - <\mathbf{v}, \mathbf{q}_i><\mathbf{q}_i, \mathbf{q}_i> \\
&= <\mathbf{v}, \mathbf{q}_i> - <\mathbf{v}, \mathbf{q}_i> \\
&= 0
\end{aligned}$$

bears this put. (**12 points**)

(b) Show that $\mathbf{u}$ is a closest vector in $S$ to $\mathbf{v}$.

**You may use**: If $\mathbf{u}, \mathbf{v} \in V$ and $<\mathbf{u}, \mathbf{v}> = 0$ then $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.

*Solution*: Let $\mathbf{u}' \in S$. Then $\mathbf{u} - \mathbf{u}' \in S$ since $S$ is a subspace of $V$. Since $(\mathbf{v} - \mathbf{u}) \perp S$ by part (a) we calculate

$$\begin{aligned} ||\mathbf{v} - \mathbf{u}||^2 &= ||(\mathbf{v} - \mathbf{u}) + (\mathbf{u} - \mathbf{u}')||^2 \\ &= ||\mathbf{v} - \mathbf{u}||^2 + ||\mathbf{u} - \mathbf{u}'||^2 \\ &\geq ||\mathbf{v} - \mathbf{u}||^2 \end{aligned}$$

which implies $||\mathbf{v} - \mathbf{u}'||^2 \geq ||\mathbf{v} - \mathbf{u}||^2$ or equivalently $||\mathbf{v} - \mathbf{u}'|| \geq ||\mathbf{v} - \mathbf{u}||$. (**13 points**)

3. Consider $\mathbf{R}^3$ as an inner product space with the standard inner product and let $S$ be the subspace with basis $\{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\}$.

(a) Find the vector in $S$ closest to $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

*Solution*: The basis vectors are perpendicular. Applying the Gram–Schmidt process to this basis yields the orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2\}$, where $\mathbf{q}_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{q}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. The closest vector in $S$ to $\mathbf{b}$ is

$$<\mathbf{b}, \mathbf{q}_1>\mathbf{q}_1 + <\mathbf{b}, \mathbf{q}_2>\mathbf{q}_2 = \frac{1}{6}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}. \quad (\textbf{8 points})$$

(b) Find a vector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2$ which is a least squares solution to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 2 & -1 \end{pmatrix}$.

*Solution*: By the previous calculation $\begin{pmatrix} 1/6 \\ -2/3 \end{pmatrix}$ (**8 points**)

(c) Let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the orthogonal projection of $\mathbf{R}^3$ onto $S$. Find a $3\times 3$ matrix $A$ such that $T(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbf{R}^3$.

*Solution*: The matrix $A = \mathbf{q}_1\mathbf{q}_1^t + \mathbf{q}_2\mathbf{q}_2^t = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. (**9 points**)

4. Find the matrix of the rotation $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ which satisfies $T\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$.

*Solution*: Here is a solution which is not geometric in nature. Since $T$ is an isometry the matrix of $T$ is $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ or $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, where $a^2 + b^2 = 1$. Since $T$ is a rotation the determinant of the matrix of $T$ is 1. Thus the first matrix is the matrix of $T$. Since $T\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ necessarily $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$. Inverting the matrix of $T$, or solving the implicit linear system directly, yields $a = 4/5$ and $b = 3/5$. Thus the matrix of $T$ is $\begin{pmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{pmatrix}$. (**25 points**)

5. Find the spectral decomposition of $A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$, given $c_A(x) = -(x+2)^2(x-4)$.

*Solution*: For $\lambda = 4$ the space of eigenvalues has basis $\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\}$ and for $\lambda = -2$ the space of eigenvalues has basis $\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\}$. Applying the Gram-Schmidt process to these two bases yields $\{\mathbf{q}_1\}$ and $\{\mathbf{q}_2, \mathbf{q}_3\}$ respectively, where $\mathbf{q}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and hence $E_1 = \mathbf{q}_1\mathbf{q}_1^t = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ (**10 points**), and $\mathbf{q}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{q}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, and therefore $E_2 = \mathbf{q}_2\mathbf{q}_2^t + \mathbf{q}_3\mathbf{q}_3^t = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ (**10 points**). Thus $A = 4E_1 + (-2)E_2$ (**5 points**).

*Comment*: Since $I_3 = E_1 + E_2$ once $E_1$ is calculated $E_2$ follows directly.

6. The only complex eigenvalue which $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ -1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 \end{pmatrix}$ has is $\lambda = 0$. Find an invertible matrix $S$ and Jordan matrix $J$ such that $A = SJS^{-1}$.

*Solution*: Regard $A$ as a matrix with complex coefficients. Since the characteristic polynomial over $\mathbf{C}$ has one root $\lambda = 0$ it follows that $c_A(x) = \pm x^n \in \mathbf{C}[x]$. Thus $A$ is nilpotent.

Note that $\mathbf{v}_2 = \mathbf{e}_1 \xrightarrow{A} \mathbf{e}_1 - \mathbf{e}_3 = \mathbf{v}_1 \xrightarrow{A} \mathbf{0}$ (**8 points**) and $\mathbf{v}_4 = \mathbf{e}_2 \xrightarrow{A} 2\mathbf{e}_2 - 2\mathbf{e}_4 = \mathbf{v}_3 \xrightarrow{A} \mathbf{0}$ (**8 points**). Set

$$S = (\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix} \qquad J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\textbf{9 points})$$

7. $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. Find the characteristic polynomial of $A$, an invertible matrix $S$, and Jordan matrix $J$ such that $A = SJS^{-1}$.

*Solution*: Expanding on the second and then the third row

$$\begin{aligned} c_A(x) &= \begin{vmatrix} 1-x & 2 & 3 & 4 \\ 0 & -x & 0 & 0 \\ 1 & 2 & 3-x & 0 \\ 0 & 1 & 0 & -x \end{vmatrix} \\ &= (-x)\begin{vmatrix} 1-x & 3 & 4 \\ 1 & 3-x & 0 \\ 0 & 0 & -x \end{vmatrix} \\ &= (-x)^2 \begin{vmatrix} 1-x & 3 \\ 1 & 3-x \end{vmatrix} \\ &= x^2((1-x)(3-x)-3) \\ &= x^2(x^2-4x) \\ &= x^3(x-4).\ (\textbf{8 points}) \end{aligned}$$

$\mathcal{N}(A - 4I_4)$ has basis $\{\mathbf{v}_1\}$, where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. Now $\mathcal{N}(A^3) = \mathcal{N}((A - 0I_4)^3) = \mathcal{R}(A - 4I_4)$ which is spanned by the columns of $A - 4I_4 = \begin{pmatrix} -3 & 2 & 3 & 4 \\ 0 & -4 & 0 & 0 \\ 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & -4 \end{pmatrix}$. Observe that

$$\mathbf{v}_2 = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{A} \mathbf{0}, \qquad \mathbf{v}_4 = \begin{pmatrix} 2 \\ -4 \\ 2 \\ 1 \end{pmatrix} \xrightarrow{A} \begin{pmatrix} 4 \\ 0 \\ 4 \\ -4 \end{pmatrix} = \mathbf{v}_3 \xrightarrow{A} \mathbf{0}.$$

Take

$$S = (\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4) = \begin{pmatrix} 1 & -3 & 4 & 2 \\ 0 & 0 & 0 & -4 \\ 1 & 1 & 4 & 2 \\ 0 & 0 & -4 & 1 \end{pmatrix} \quad (\textbf{12 points}) \qquad J = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (\textbf{5 points})$$

8. Suppose $\mathbf{v} \in V$, $n > 0$ and $T^n(\mathbf{v}) = \mathbf{0} \neq T^{n-1}(\mathbf{v})$.

(a) Show that $\{\mathbf{v}, T(\mathbf{v}), \ldots, T^{n-1}(\mathbf{v})\}$ is linearly independent.

*Solution*: Here is a detailed argument. First of all note for $m \geq n$ that

$$T^m(\mathbf{v}) = \mathbf{0} \tag{1}$$

as $T^m(\mathbf{v}) = T^{n-m}(T^m(\mathbf{v})) = T^{n-m}(\mathbf{0}) = \mathbf{0}$.

Let $a_0, \ldots, a_{n-1} \in \mathbf{R}$ and suppose that

$$a_0\mathbf{v} + a_1T(\mathbf{v}) + \cdots + a_{n-1}T^{n-1}(\mathbf{v}) = \mathbf{0}. \tag{2}$$

We will show that $a_0 = \cdots = a_\ell = 0$ for all $0 \leq \ell \leq n-1$. Thus $a_0 = \cdots = a_{n-1} = 0$ which means that $\{\mathbf{v}, T(\mathbf{v}), \ldots, T^{n-1}(\mathbf{v})\}$ is linearly independent.

Applying $T^{n-1}$ to both sides of (2) yields $a_0T^{n-1}(\mathbf{v}) = \mathbf{0}$ by (1). Since $T^{n-1}(\mathbf{v}) \neq \mathbf{0}$ by assumption $a_0 = 0$. (**3 points**)

Now suppose that $0 \leq \ell < n-1$ and $a_0 = \cdots = a_\ell = 0$. Since $n - \ell - 1 > 0$ we may apply $T^{n-\ell-1}$ to both sides of (2) which yields $a_0T^{n-\ell-1}(\mathbf{v}) + \cdots + a_\ell T^{n-1}(\mathbf{v}) = \mathbf{0}$ or equivalently $a_\ell T^{n-1}(\mathbf{v}) = \mathbf{0}$. As $T^{n-1}(\mathbf{v}) \neq \mathbf{0}$ by assumption $a_\ell = 0$. (**5 points**)

*Comment*: The preceding argument can be turned into a formal induction argument. Let $P_\ell$ be the statement $a_0, \ldots, a_\ell = 0$ for $0 \leq \ell \leq n-1$ and let $P_\ell$ be any true statement for $n \leq \ell$.

(b) Show that $\{I, T, \ldots, T^{n-1}\}$ is linearly independent.

*Solution*: Suppose that $a_0, \ldots, a_{n-1} \in \mathbf{R}$ and $a_0I + a_1T + \cdots + a_{n-1}T^{n-1} = 0$. Applying both sides of this equation to $\mathbf{v}$ yields $a_0\mathbf{v} + a_1T(\mathbf{v}) + \cdots + a_{n-1}T^{n-1}(\mathbf{v}) = \mathbf{0}$. Since $\{\mathbf{v}, T(\mathbf{v}), \ldots, T^{n-1}(\mathbf{v})\}$ is linearly independent by part (a) necessarily $a_0 = \cdots = a_{n-1} = 0$. Thus $\{I, T, \ldots, T^{n-1}\}$ is linearly independent. (**4 points**)

(c) Suppose that $T^n = 0$. Show that $m_T(x) = x^n$.

*Solution*: $\{I, T, \ldots, T^{n-1}\}$ is linearly independent (**3 points**) by part (b) and $\{I, T, \ldots, T^n\}$ is linearly dependent (**3 points**) since $T^n = 0$. Since $0I + \cdots + 0T^{n-1} + 1T^n = 0$ by definition (**4 points**) of the minimal polynomial $m_T(x) = 0 + 0x + \cdots + 0x^{n-1} + x^n = x^n$. (**3 points**)