MATH 425 **Hour Exam II Solution** **Radford** 18/04/08

1. (**20**) The characteristic polynomial of of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ is $c_A(x) = 3 - 4x + x^2 = (x-1)(x-3)$. Thus $\lambda = 1, 3$ are the eigenvalues of $A$. The corresponding eigenspaces have bases $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$ and $\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$ respectively. Applying the Gram-Schmidt process gives orthonormal bases $\{q_1\}$ and $\{q_2\}$ respectively, where $q_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $q_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus $E_1 = q_1 q_1^t = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ (**7 points**) and $E_2 = q_2 q_2^t = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (**7 points**). $A = 1E_1 + 3E_2$ (**6 points**).

2. (**20**) $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \perp \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and therefore $T\begin{pmatrix} 1 \\ 2 \end{pmatrix} \perp T\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Since $T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ it follows that $T\begin{pmatrix} 2 \\ -1 \end{pmatrix} = r\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ for some $r \in \mathbf{R}$. Since

$$|r|\left\|\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\| = \left\|r\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\| = \left\|T\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right\| = \left\|\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right\|$$

it follows that $|r|\sqrt{5} = \sqrt{5}$ and thus $r = \pm 1$ (**9 points**). THERE ARE TWO CASES, ONLY ONE IS NECESSARY TO WORK OUT.

*Case* 1: $r = 1$. Using the hint we calculate

$$\begin{aligned} T\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{x+2y}{5}T\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{2x-y}{5}T\begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &= \frac{x+2y}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{2x-y}{5}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 4x+3y \\ -3x+4y \end{pmatrix} \quad (\textbf{6 points}) \\ &= \frac{1}{5}\begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

Since the matrix of $T$ has determinant 1, it follows that $T$ is a rotation (**5 points**).

*Case* 2: $r = -1$. Similar calculations yield $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Since the matrix of $T$ has determinant $-1$, it follows that $T$ is a reflection.

3. (**20**) We apply the Gram-Schmidt process to the basis $\{1, x\}$ for $\mathbf{P}^1$ to get the orthonormal basis $\{q_1, q_2\}$, where $q_1 = \frac{1}{\sqrt{2}}$ (**4 points**), $q_2 = \sqrt{\frac{3}{2}}(x-1)$ (**4 points**). Since $\{e_1, e_2\}$ and $\{q_1, q_2\}$ are bases for $\mathbf{R}^2$ and $\mathbf{P}^1$ respectively, by the hint there is a

linear isomorphism $f : \mathbf{R}^2 \longrightarrow \mathbf{P}^1$ determined by $f(e_1) = q_1$ and $f(e_2) = q_2$. Now $f$ is an isometry of inner product spaces since these are orthonormal bases. (**5 points**).

Now $f\begin{pmatrix} a \\ b \end{pmatrix} = f(ae_1 + be_2) = af(e_1) + bf(e_2) = a\frac{1}{\sqrt{2}} + b\sqrt{\frac{3}{2}}(x-1)$ (**7 points**).

4. (**20**) Here is a detailed solution, more than what was necessary. The problem implies that $A = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}$ is nilpotent since it is similar to a nilpotent matrix (the two matrices have the same characteristic polynomial).

Let $v_3 = e_1$, $v_2 = Av_3 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$, and $v_1 = Av_2 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$. Since $Av_1 = 0$, the set of non-zero vectors $\{v_1, v_2, v_3\}$ forms an $A$-string which is thus independent and a basis for $\mathbf{R}^3$. Thus $S = (v_1\ v_2\ v_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 3 & 2 & 0 \end{pmatrix}$ (**20 points**).

5. (**20**) Here is a very detailed solution, more than what was necessary. $A = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 1 & 3 \end{pmatrix}$ has characteristic polynomial $c_A(x) = (3-x)^2(4-x)$. Thus $\mathrm{Dim}\,\mathcal{N}(A - 4\mathrm{I}_3) = 1$, $\mathrm{Dim}\,\mathcal{N}((A - 3\mathrm{I}_3)^2) = 2$, and $\mathcal{R}(A - 4\mathrm{I}_3) = \mathcal{N}((A - 3\mathrm{I}_3)^2)$.

A basis for $\mathcal{N}(A - 4\mathrm{I}_3)$, the space of eigenvectors for $A$ belonging to $\lambda = 4$, consists of the vector $v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ (**4 points**). Let $v_3 = (A - 4\mathrm{I}_3)e_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ (**6 points**) and $v_2 = (A - 3\mathrm{I}_3)v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (**4 points**). Since $(A - 3\mathrm{I}_3)v_2 = 0$, and $v_2, v_3 \neq 0$, it follows that $v_2$ and $v_3$ form a basis for $\mathcal{N}((A - 3\mathrm{I}_3)^2)$. Therefore $\{v_1, v_2, v_3\}$ form a basis for $\mathbf{R}^3$ and $S = (v_1\ v_2\ v_3) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ (**6 points**).