MATH 425 **Hour Exam I Solution** **Radford** **02/22/08**

1. (**20 points**) First

(a) observe that $a+bx+cx^2 \in S^\perp$ if and only if $0 = <a+bx+cx^2,\ x> = \int_0^2 (a+bx+cx^2)x\,dx$
$= \int_0^2 (ax+bx^2+cx^3)\,dx = (a\frac{x^2}{2}+b\frac{x^3}{3}+c\frac{x^4}{4})\Big|_0^2 = 2a+\frac{8}{3}b+4c$; $0 = a+\frac{4}{3}b+2c$. (**10**)

(b) Solutions to this equation, which is equivalent to the system $\begin{array}{rcrr} a & = & -\frac{4}{3}b & -2c \\ b & = & b & \\ c & = & & c \end{array}$, is given by $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = b\begin{pmatrix} -4/3 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$. A basis for $S^\perp$ is $\{-\frac{4}{3}+x,\ -2+x^2\}$. (**10**)

2. (**20 points**) Recall that $\mathbf{s}'$ is a closest vector in S to $\mathbf{v}$ means $\mathbf{s} \in S$ and $||\mathbf{v}-\mathbf{s}'|| \leq ||\mathbf{v}-\mathbf{s}||$ for all $\mathbf{s} \in S$.

(a) Let $\mathbf{s} \in S$. Then

$$||\mathbf{v}-\mathbf{s}||^2 = ||(\mathbf{v}-\mathbf{s}_0)+(\mathbf{s}_0-\mathbf{s})||^2 = ||\mathbf{v}-\mathbf{s}_0||^2 + ||\mathbf{s}_0-\mathbf{s}||^2 \geq ||\mathbf{v}-\mathbf{s}_0||^2, \qquad (1)$$

where the second equation follows since $\mathbf{s}_0 - \mathbf{s} \in S$. By (1) we deduce $||\mathbf{v}-\mathbf{s}||^2 \geq ||\mathbf{v}-\mathbf{s}_0||^2$, hence $||\mathbf{v}-\mathbf{s}|| \geq ||\mathbf{v}-\mathbf{s}_0||$. (**10**)

(b) $||\mathbf{v}-\mathbf{s}|| \geq ||\mathbf{v}-\mathbf{s}_1||$ for all $\mathbf{s} \in S$; in particular for $\mathbf{s} = \mathbf{s}_0$. Thus by (1), with $\mathbf{s} = \mathbf{s}_1$, we deduce $||\mathbf{v}-\mathbf{s}_0||^2 \geq ||\mathbf{v}-\mathbf{s}_1||^2 = ||\mathbf{v}-\mathbf{s}_0||^2 + ||\mathbf{s}_0-\mathbf{s}_1||^2 \geq ||\mathbf{v}-\mathbf{s}_0||^2$, from which $||\mathbf{v}-\mathbf{s}_0||^2 = ||\mathbf{v}-\mathbf{s}_0||^2 + ||\mathbf{s}_0-\mathbf{s}_1||^2$, or equivalently $||\mathbf{s}_0-\mathbf{s}_1||^2 = 0$, follows. Therefore $||\mathbf{s}_0-\mathbf{s}_1|| = 0$ and consequently $\mathbf{s}_0 = \mathbf{s}_1$. (**10**)

3. (**20 points**) Let $\{\mathbf{q}_1, \mathbf{q}_2\}$ be the orthonormal basis, and let $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

(a) By inspection $<\mathbf{v},\ \mathbf{q}_1> = -\frac{2}{13}$ and $<\mathbf{v},\ \mathbf{q}_2> = \frac{7}{13}$. Thus the closest vector is

$$<\mathbf{v},\ \mathbf{q}_1>\mathbf{q}_1 + <\mathbf{v},\ \mathbf{q}_2>\mathbf{q}_2 = \frac{1}{169}\begin{pmatrix} -6-28 \\ 24+0 \\ -8+21 \\ 0+84 \end{pmatrix} = \frac{1}{169}\begin{pmatrix} -34 \\ 24 \\ 13 \\ 84 \end{pmatrix}.$$

(**10**)

(b) $A = \mathbf{q}_1\mathbf{q}_1^t + \mathbf{q}_2\mathbf{q}_2^t = \frac{1}{169}\left(\begin{pmatrix} 3 \\ -12 \\ 4 \\ 0 \end{pmatrix}(3 \quad -12 \;\; 4 \;\; 0) + \begin{pmatrix} -4 \\ 0 \\ 3 \\ 12 \end{pmatrix}(-4 \;\; 0 \;\; 3 \;\; 12)\right)$

$$= \frac{1}{169}\left(\begin{pmatrix} 9 & -36 & 12 & 0 \\ -36 & 144 & -48 & 0 \\ 12 & -48 & 16 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 16 & 0 & -12 & -48 \\ 0 & 0 & 0 & 0 \\ -12 & 0 & 9 & 36 \\ -48 & 0 & 36 & 144 \end{pmatrix}\right)$$

$$= \frac{1}{169}\begin{pmatrix} 25 & -36 & 0 & -48 \\ -36 & 144 & -48 & 0 \\ 0 & -48 & 25 & 36 \\ -48 & 0 & 36 & 144 \end{pmatrix}.(\mathbf{10})$$

4. (**20 points**) The characteristic polynomial of A is $c_A(x) = \begin{vmatrix} 3-x & 8 & 4 \\ 0 & 5-x & 1 \\ 0 & 0 & 3-x \end{vmatrix} =$ $(3-x)(5-x)(3-x)$. Thus the eigenvalues for A are $\lambda = 3, 5$.

$\lambda = 3$: $A - \lambda I_3 = \begin{pmatrix} 0 & 8 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 0 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; eigenvectors in vector form are $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}$. Thus $\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}\}$ is a basis for the space of eigenvectors for A belonging to $\lambda = 3$.

$\lambda = 5$: $A - \lambda I_3 = \begin{pmatrix} -2 & 8 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} -2 & 8 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 1 & -4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$; eigenvectors in vector form are $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$. Thus $\{\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}\}$ is a basis for the space of eigenvectors for A belonging to $\lambda = 5$.

Take $S = \begin{pmatrix} 1 & -1/2 & 4 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ (**10**) and $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ (**10**).

5. (**20 points**) Note that A is a transition matrix.

(a) This is the nullspace of $A - I_3$. By row redutcion $A - I_3 = \begin{pmatrix} -1/2 & 1/2 & 1/4 \\ 1/4 & -3/4 & 1/4 \\ 1/4 & 1/4 & -1/2 \end{pmatrix} \longrightarrow$

$$\cdots \longrightarrow \begin{pmatrix} -2 & 2 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -2 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 0 & 4 & -3 \\ 0 & -4 & 3 \\ 1 & 1 & 2 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -3/4 \\ 1 & 0 & -5/4 \end{pmatrix}$$

which means $\{\begin{pmatrix} 5/4 \\ 4/3 \\ 1 \end{pmatrix}\}$, or $\{\begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}\}$ is a basis for the set of solutions. (**10**)

(b) Since A is a transition matrix has a row of non-zero entries it follows that A has a unique probability distribution. Such vectors are probability vectors. Thus from part (a) the vector we seek is $\begin{pmatrix} 15/12 \\ 1/4 \\ 1/3 \end{pmatrix}$. (**10**)