The Group of Even Permutations A_n

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Let $n \geq 2$. We will show that A_n is simple for $n \neq 4$ and show that A_4 has a unique normal subgroup which lies properly between (e) and A_4. Recall that all 3-cycles are even permutations.

Suppose that $\tau, \sigma \in S_n$. We write ${}^{\tau}\sigma = \tau\sigma\tau^{-1}$. Since conjugation is an automorphism of S_n the formula

$$ {}^{\tau}(\sigma_1 \cdots \sigma_r) = {}^{\tau}\sigma_1 \cdots {}^{\tau}\sigma_r \tag{1} $$

holds for all $\sigma_1, \ldots, \sigma_r \in S_n$. Note that

$$ {}^{\tau}(a_1 \ \ldots \ a_r) = (\tau(a_1) \ \ldots \ \tau(a_r)) \tag{2} $$

holds for all r-cycles in S_n. In particular if $\sigma = \sigma_1 \cdots \sigma_r$ is a decomposition of σ into disjoint cycles then ${}^{\tau}(\sigma) = {}^{\tau}\sigma_1 \cdots {}^{\tau}\sigma_r$ is a decomposition of ${}^{\tau}\sigma$ into disjoint cycles.

Lemma 1 *Let $n \geq 3$. Then A_n is generated by 3-cycles.*

PROOF: All 3-cycles belong to A_n since they are even permutations. Let $\sigma \in A_n$. Then $\sigma = \tau_1\tau_2 \cdots \tau_{2r}$ is the product of an even number of transpositions. Thus we may write $\sigma = (\tau_1\tau_2) \cdots (\tau_{2r-1}\tau_{2r})$. Now a product of transpositions $\tau\tau'$ has the form $(a \ b)(a \ b)$, $(a \ b)(a \ c)$, or $(a \ b)(c \ d)$, where the symbols a, b, c, d are distinct. The calculations

$$ (a\ b)(a\ b) = \mathrm{Id}, \quad (a\ b)(a\ c) = (a\ c\ b), \quad \text{and} \quad (a\ b)(c\ d) = (a\ b\ c)(b\ c\ d) $$

show that σ is the product of 3-cycles. $\Box$

Lemma 2 *Suppose $N \trianglelefteq A_n$ and contains a 3-cycle. Then $N = A_n$.*

Proof: Suppose that $\sigma = (a\ b\ c) \in N$. Then $S = \{a, b, c\}$ is the unique non-trivial orbit of σ. The 3-cycles which have S as their non-trivial orbit are $(a\ b\ c)$ and $(a\ c\ b) = (a\ b\ c)^{-1}$. In light of Lemma 1 we need only show that any 3-element subset S' of $\{1, \ldots, n\}$ is the non-trivial orbit of a 3-cycle in N. We do this in three cases. We may assume $S \neq S'$, or equivalently $|S \cap S'| < 3$.

Case 1: $|S \cap S'| = 2$. We may assume that $S' = \{a, b, d\}$, where $d \notin S$. Since $(a\ b)(c\ d) \in A_n$,

$$(b\ a\ d) = {}^{(a\ b)(c\ d)}(a\ b\ c) \in N.$$

Case 2: $|S \cap S'| = 1$. We may assume that $S' = \{a, d, e\}$ where $d, e \notin S$. Since $(b\ d)(c\ e) \in A_n$,

$$(a\ d\ e) = {}^{(b\ d)(c\ e)}(a\ b\ c) \in N.$$

Case 3: $|S \cap S'| = 0$. Then $S' = \{d, e, f\}$ where $d, e, f \notin S$. Since $(a\ d\ b\ e)(c\ f) \in A_n$,

$$(d\ e\ f) = {}^{(a\ d\ b\ e)(c\ f)}(a\ b\ c) \in N.$$

This completes our proof. □

Lemma 3 *Suppose that* $n \geq 3$ *and* $(\mathrm{Id}) \neq N \trianglelefteq A_n$. *Then* N *contains a product of two disjoint transpositions or a* 3*-cycle.*

Proof: By assumption there is permutation in N which is not the identity. Among these permutations choose one σ which has the most fixed points (the most one-element orbits) and consider its decomposition into disjoint cycles.

Suppose that σ has a cycle $(a\ b\ c\ \cdots\ d)$ of length at least 4. Then

$$\begin{aligned} ({}^{(a\ b\ d)}\sigma)\sigma^{-1} &= \left({}^{(a\ b\ d)}(a\ b\ c\ \ldots\ d)\right)(a\ b\ c\ \ldots\ d)^{-1} \\ &= (b\ d\ c\ \ldots\ a)(d\ \ldots\ c\ b\ a) \\ &= (b)(a\ c\ d\ \ldots) \end{aligned}$$

belongs to N, is not the identity, and has more fixed points than σ. This contradiction shows that σ is the product of disjoint cycles which have length

2 or 3. We will show that σ is the product of two disjoint 2-cycles or σ is a 3-cycle.

Case 1: σ is the product of disjoint 2-cycles.

We may write $\sigma = (a\ b)(c\ d)\cdots$. Since

$$\begin{aligned} ({}^{(a\ b\ c)}\sigma)\sigma^{-1} &= \left({}^{(a\ b\ c)}((a\ b)(c\ d))\right)((a\ b)(c\ d))^{-1} \\ &= (b\ c)(a\ d)(a\ b)(c\ d) = (a\ c)(b\ d) \end{aligned}$$

belongs to N and fixes all points except four, σ fixes all but at most four points by our choice of σ. Therefore σ is the product two disjoint 2-cycles.

Case 2: One of the cycles of σ is a 3-cycle.

In this case $\sigma^2 \in N$, is not Id, is the product of disjoint 3-cycles, and has at least as many fixed points as σ. Suppose $\sigma^2 = (a\ b\ c)(d\ e\ f)\cdots$. Then

$$\begin{aligned} \left({}^{(a\ b\ d)}(\sigma^2)\right)\sigma^{-2} &= \left({}^{(a\ b\ d)}((a\ b\ c)(d\ e\ f))\right)((a\ b\ c)(d\ e\ f))^{-1} \\ &= (b\ d\ c)(a\ e\ f)(a\ c\ b)(d\ f\ e) \\ &= (a\ b\ e\ c\ d)(f) \end{aligned}$$

belongs to N and fixes fewer points than σ, a contradiction. Therefore σ^2 is a 3-cycle which means σ is a 3-cycle. $\Box$

Theorem 1 *Let $n \geq 2$. Then:*

(a) *A_n is simple if $n = 2, 3$ or $n \geq 5$.*

(b) *A_n is not simple when $n = 4$. The normal subgroups of A_4 are* (Id), *A_4, and $N = \{\mathrm{Id}, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$.*

PROOF: First of all A_n is simple if $n = 2, 3$ since $A_n \simeq \mathbf{Z}_n$ in these cases. Let $n \geq 4$ and suppose $N \trianglelefteq A_n$ satisfy $N \neq (\mathrm{Id}), A_n$. Then N does not contain a 3-cycle by Lemma 2. By Lemma 3 N contains a product of two disjoint 2-cycles $(a\ b)(c\ d)$. The calculation

$$\left({}^{(a\ b\ e)}((a\ b)(c\ d))\right)((a\ b)(c\ d))^{-1} = (b\ e)(c\ d)(a\ b)(c\ d) = (a\ e\ b)$$

shows that $n \not\geq 5$. Therefore $n = 4$. Since

$${}^{(a\ b\ c)}((a\ b)(c\ d)) = (b\ c)(a\ d) = (a\ d)(b\ c)$$

and

$$^{(a\ b\ c)}((a\ d)(b\ c)) = (b\ d)(c\ a) = (a\ c)(b\ d)$$

it follows that N contains the subgroup of part (b). Since N has no 3-cycles, N must be the subgroup of part (b). Using (1) it is easy to see that, in fact, $N \trianglelefteq S_n$.□