Let $n \geq 2$. We will show that $A_n$ is simple for $n \neq 4$ and show that $A_4$ has a unique normal subgroup which lies properly between $(e)$ and $A_4$. Recall that all 3-cycles are even permutations.

Suppose that $\tau, \sigma \in S_n$. We write $\tau \sigma = \tau \sigma \tau^{-1}$. Since conjugation is an automorphism of $S_n$ the formula

$$\tau(\sigma_1 \cdots \sigma_r) = \tau \sigma_1 \cdots \tau \sigma_r$$

holds for all $\sigma_1, \ldots, \sigma_r \in S_n$. Note that

$$\tau(a_1 \ldots a_r) = (\tau(a_1) \ldots \tau(a_r))$$

holds for all $r$-cycles in $S_n$. In particular if $\sigma = \sigma_1 \cdots \sigma_r$ is a decomposition of $\sigma$ into disjoint cycles then $\tau(\sigma) = \tau \sigma_1 \cdots \tau \sigma_r$ is a decomposition of $\tau \sigma$ into disjoint cycles.

**Lemma 1** Let $n \geq 3$. Then $A_n$ is generated by 3-cycles.

**Proof:** All 3-cycles belong to $A_n$ since they are even permutations. Let $\sigma \in A_n$. Then $\sigma = \tau_1 \tau_2 \cdots \tau_{2r}$ is the product of an even number of transpositions. Thus we may write $\sigma = (\tau_1 \tau_2) \cdots (\tau_{2r-1} \tau_{2r})$. Now a product of transpositions $\tau \tau'$ has the form $(a \ b)(a \ b)$, $(a \ b)(a \ c)$, or $(a \ b)(c \ d)$, where the symbols $a, b, c, d$ are distinct. The calculations

$$(a \ b)(a \ b) = \text{Id}, \quad (a \ b)(a \ c) = (a \ c \ b), \quad \text{and} \quad (a \ b)(c \ d) = (a \ b \ c)(b \ c \ d)$$

show that $\sigma$ is the product of 3-cycles. $\square$

**Lemma 2** Suppose $N \triangleleft A_n$ and contains a 3-cycle. Then $N = A_n$. 

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Proof: Suppose that \( \sigma = (a \ b \ c) \in N \). Then \( S = \{a, b, c\} \) is the unique non-trivial orbit of \( \sigma \). The 3-cycles which have \( S \) as their non-trivial orbit are \((a \ c \ b) = (a \ b \ c)^{-1}\). In light of Lemma 1 we need only show that any 3-element subset \( S' \) of \( \{1, \ldots, n\} \) is the non-trivial orbit of a 3-cycle in \( N \). We do this in three cases. We may assume \( S \neq S' \), or equivalently \(|S \cap S'| < 3|\).

Case 1: \(|S \cap S'| = 2\). We may assume that \( S' = \{a, b, d\} \), where \( d \not\in S \). Since \((a \ b)(c \ d) \in A_n\),

\[
(b \ a \ d) = (a \ b)(c \ d)(a \ b \ c) \in N.
\]

Case 2: \(|S \cap S'| = 1\). We may assume that \( S' = \{a, d, e\} \) where \( d, e \not\in S \). Since \((b \ d)(c \ e) \in A_n\),

\[
(a \ d \ e) = (b \ d)(c \ e)(a \ b \ c) \in N.
\]

Case 3: \(|S \cap S'| = 0\). Then \( S' = \{d, e, f\} \) where \( d, e, f \not\in S \). Since \((a \ d \ b \ e)(c \ f) \in A_n\),

\[
(d \ e \ f) = (a \ d \ b \ e)(c \ f)(a \ b \ c) \in N.
\]

This completes our proof. \( \Box \)

Lemma 3 Suppose that \( n \geq 3 \) and \((\text{Id}) \neq N \leq A_n\). Then \( N \) contains a product of two disjoint transpositions or a 3-cycle.

Proof: By assumption there is permutation in \( N \) which is not the identity. Among these permutations choose one \( \sigma \) which has the most fixed points (the most one-element orbits) and consider its decomposition into disjoint cycles.

Suppose that \( \sigma \) has a cycle \((a \ b \ c \ \ldots \ d)\) of length at least 4. Then

\[
((a \ b \ d) \sigma)\sigma^{-1} = (a \ b \ d)(a \ b \ c \ \ldots \ d)(a \ b \ c \ \ldots \ d)^{-1} = (b \ d \ c \ \ldots \ a)(d \ \ldots \ c \ b \ a) = (b)(a \ c \ d \ \ldots)
\]

belongs to \( N \), is not the identity, and has more fixed points than \( \sigma \). This contradiction shows that \( \sigma \) is the product of disjoint cycles which have length
2 or 3. We will show that $\sigma$ is the product of two disjoint 2-cycles or $\sigma$ is a 3-cycle.

**Case 1:** $\sigma$ is the product of disjoint 2-cycles.

We may write $\sigma = (a \ b)(c \ d) \cdots$. Since

\[
\left( (a \ b \ c)\sigma \right)^{-1} = \left( (a \ b \ c)\left((a \ b)(c \ d)\right)\right) \left((a \ b)(c \ d)\right)^{-1} \\
= \left( b \ c \right)(a \ d)(a \ b)(c \ d) = (a \ c)(b \ d)
\]

belongs to $N$ and fixes all points except four, $\sigma$ fixes all but at most four points by our choice of $\sigma$. Therefore $\sigma$ is the product two disjoint 2-cycles.

**Case 2:** One of the cycles of $\sigma$ is a 3-cycle.

In this case $\sigma^2 \in N$, is not Id, is the product of disjoint 3-cycles, and has at least as many fixed points as $\sigma$. Suppose $\sigma^2 = (a \ b \ c)(d \ e \ f) \cdots$. Then

\[
\left( (a \ b \ d)\left(\sigma^2\right)\right)^{-2} = \left( (a \ b \ d)\left((a \ b \ c)(d \ e \ f)\right)\right) \left((a \ b \ c)(d \ e \ f)\right)^{-1} \\
= (b \ d \ c)(a \ e \ f)(a \ c \ b)(d \ f \ e) \\
= (a \ b \ c \ d)(f)
\]

belong to $N$ and fixes fewer points than $\sigma$, a contradiction. Therefore $\sigma^2$ is a 3-cycle which means $\sigma$ is a 3-cycle. □

**Theorem 1** Let $n \geq 2$. Then:

(a) $A_n$ is simple if $n = 2, 3$ or $n \geq 5$.

(b) $A_n$ is not simple when $n = 4$. The normal subgroups of $A_4$ are (Id), $A_4$, and $N = \{Id, (1 \ 2)(3 \ 4), (1 \ 3)(2 \ 4), (1 \ 4)(2 \ 3)\}$.

**Proof:** First of all $A_n$ is simple if $n = 2, 3$ since $A_n \simeq \mathbb{Z}_n$ in these cases. Let $n \geq 4$ and suppose $N \leq A_n$ satisfy $N \neq (Id), A_n$. Then $N$ does not contain a 3-cycle by Lemma 2. By Lemma 3 $N$ contains a product of two disjoint 2-cycles $(a \ b)(c \ d)$. The calculation

\[
\left( (a \ b \ c)\left((a \ b)(c \ d)\right)\right) \left((a \ b)(c \ d)\right)^{-1} = (b \ e)(c \ d)(a \ b)(c \ d) = (a \ e \ b)
\]

shows that $n \neq 5$. Therefore $n = 4$. Since

\[
(a \ b \ c)\left((a \ b)(c \ d)\right) = (b \ c)(a \ d) = (a \ d)(b \ c)
\]
and
\[(a \, b \, c)((a \, d)(b \, c)) = (b \, d)(c \, a) = (a \, c)(b \, d)\]
it follows that \(N\) contains the subgroup of part (b). Since \(N\) has no 3-cycles, \(N\) must be the subgroup of part (b). Using (1) it is easy to see that, in fact, \(N \leq S_n\). \(\square\)