1. Let $G$ be a group and $H, K \leq G$.

   (a) Suppose that $HK \leq G$ and let $f : H \times K \rightarrow HK$ be defined by $f((h, k)) = hk$ for all $(h, k) \in H \times K$. Show that $f$ is a homomorphism if and only if $hk = kh$ for all $h \in H$ and $k \in K$.

   Suppose in addition that $H, K \nleq G$.

   (b) Show that $HK \nleq G$.

   (c) Suppose that $H \cap K = \{e\}$. Show that $hk = kh$ for all $h \in H$ and $k \in K$ and that the homomorphism of part (b) is an isomorphism. [Hint: For $h \in H$ and $k \in K$ consider $hkh^{-1}k^{-1}$.]

2. Use the theory of finite cyclic groups and induction on $|G|$ to prove Cauchy’s Theorem for abelian groups:

   **Theorem 1** Let $G$ be a finite abelian group and suppose that $p$ is a prime integer which divides $|G|$. Then $G$ as an element of order $p$. 
[Hint: Let \( a \in G \) and set \( H = \langle a \rangle \). Then \( |G/H||H| = |G| \).]
3. Let \( G \) be a finite group. For every positive divisor \( d \) of \( |G| \) let \( n_d \) denote the number of cyclic subgroup of \( G \) of order \( d \). Show that

\[
|G| = \sum_{d|\mid |G|} \varphi(d)n_d,
\]

where \( \varphi \) is the Euler phi-function. [Hint: Consider the equivalence relation on \( G \) defined by \( a \sim b \) if and only if \( \langle a \rangle = \langle b \rangle \).]

4. Let \( G \) be a finite group of order \( pqr \), where \( p, q, r \) are primes and \( p < q < r \).

(a) Show that \( G \) is not simple.

(b) Show that \( G \) has a subgroup of prime index.

[Hint: See the text’s discussion of groups of order \( 30 = 2 \cdot 3 \cdot 5 \). If needed, you may use the formula of Exercise 3.]

5. Let \( G \) be a finite group of order \( pqr \), where \( p, q, r \) are primes, \( p < q < r \), and \( r \not\equiv 1 \pmod{q} \). Show that \( G \) has a subgroup of index \( p \).