1. Let $R$ be a ring with unity (identity). Show that every element of $R$ is either a unit or a zero divisor if

(a) $R$ is finite or

(b) $R = M_n(k)$, where $k$ is a field.

[Hint: Let $a \in R$ and consider the sequence $1, a, a^2, a^3, \ldots$, noting that its terms belong to a finite set or a finite-dimensional vector space.]

2. Let $R$ be a commutative ring with unity and let $N$ be the set of nilpotent elements of $R$.

(a) Show that $N$ is an ideal of $R$. [Hint: Let $a, b \in R$. You may assume that the binomial theorem holds for $a, b$ and that $(ab)^n = a^n b^n$ for all $n \geq 0$.]

(b) Let $U = \{1 + n \mid n \in N\}$. Show that $U \subseteq R^\times$. [Hint: Show that $U = \{1 - n \mid n \in N\}$ also. If $n^\ell = 0$ then $1 - n^\ell = 1$.]

(c) Find a ring with unity whose set of nilpotent elements is not an ideal. Justify your answer. [Hint: Consider $M_2(k)$ where $k$ is a field.]

3. Let $R$ be a commutative ring with unity and set $\mathcal{R} = R[[X]]$. 


(a) Show that \( f : \mathcal{R} \rightarrow R \) defined by \( f(\sum_{n=0}^{\infty} a_n X^n) = a_0 \) is a ring homomorphism.

(b) Show that \( \sum_{n=0}^{\infty} a_n X^n \in \mathcal{R}^\times \) if and only if \( a_0 \in R^\times \).

(c) Show that \( \mathcal{R} \) is an integral domain if and only if \( R \) is an integral domain.

4. Let \( R \) be ring with unity.

(a) Suppose that \( I \) is a non-empty family of ideals of \( R \). Show that \( J = \bigcap_{I \in \mathcal{I}} I \) is an ideal of \( R \). (Since \( R \) is an ideal of \( R \), it follows that any \( S \) subset of \( R \) is contained in a smallest ideal of \( R \), namely the intersection of all ideals containing \( S \). This ideal is denoted by \( (S) \) and is called the ideal of \( R \) generated by \( S \).)

(b) Suppose that \( R \) is commutative and \( S = \{a_1, \ldots, a_r\} \) is a finite subset of \( R \). Show that \( (S) = Ra_1 + \cdots + Ra_r \).

5. Let \( R \) by any ring with unity 1 and \( \mathcal{R} = M_n(R) \). Let \( J \) be an ideal of \( R \).

(a) Show that \( M_n(J) \) is an ideal of \( \mathcal{R} \) and all ideals of \( \mathcal{R} \) have this form.

(b) Show that \( \mathcal{R} \) is simple if and only if \( R \) is simple.

[Hint: For part (a) let \( E_{ij} \in M_n(R) \) be defined by \( (E_{ij})_{k\ell} = \delta_{i,k}\delta_{j,\ell} \), where \( \delta_{u,v} = \begin{cases} 1 : u = v \\ 0 : u \neq v \end{cases} \). Work out the formula for \( E_{ij}E_{k\ell} \). Show that any \( A = (A_{uv}) \in M_n(R) \) can be written \( A = \sum_{u,v=1}^{n} A_{uv} E_{uv} \) and consider \( E_{ij}AE_{k\ell} \).]