MATH 516 **Midterm** **Radford** **WEDNESDAY 10/24/01**

(1) *Return* this exam copy with your exam booklet. (2) *Write* your solutions in your exam booklet. (3) *Show* your work. (4) There are *five questions* on this exam. (5) Each problem counts 20 points. (6) *You are expected to abide by the University's rules concerning academic honesty.*

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1. Suppose that G is a finite group with subgroups H, K. Prove that if $|H|$ and $|K|$ are relatively prime then $H \cap K = (e)$.

2. Let $f : G \longrightarrow G$ be a group homomorphism, let S be a non-empty subset of G and suppose that $f(S) \subseteq {<}S{>}$. Show that $f({<}S{>}) \subseteq {<}f(S){>}$.

3. The group $\mathrm{GL}_2(\mathbf{R})$ of 2×2 invertible matrices with coefficients in the real numbers $\mathbf{R}$ acts on $A = \mathbf{R}^2$ by $g{\cdot}\mathbf{v} = g\mathbf{v}$ for $g \in G$ and $\mathbf{v} \in A$. Let $G = \{\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \mid b \in \mathbf{Q}\}$.

   a) Show that G is a subgroup of $\mathrm{GL}_2(\mathbf{R})$.

   b) Show that the G-orbits of A have the form $\mathcal{L}_y = \{\begin{pmatrix} 0 \\ y \end{pmatrix}\}$, where $y \in \mathbf{R}$, or $\mathcal{U}_x = \{\begin{pmatrix} x \\ y \end{pmatrix} \mid y \in \mathbf{R}\}$, where $x \in \mathbf{R}\backslash 0$.

4. Let $G = S_6$, $\sigma = (1\,3\,2\,5)(4\,1\,6) \in G$ and $H = {<}\sigma{>}$.

   a) List all of the elements of H as products of disjoint cycles.

   b) Show that H is not a normal subgroup of G. [Hint: Consider $\tau\sigma\tau^{-1}$, where τ has order 3.]

   c) Let $\tau = (1\,5)(2\,6)(3\,4)$. Show that $\tau\sigma\tau^{-1} \in H$. [Thus $\tau \in \mathrm{N}_G(H)$ by Problem 2.]

   d) Show that 12 divides $|\mathrm{N}_G(H)|$.

5. Let G be a group of order $3{\cdot}5{\cdot}19$.

   a) Show that G has a unique subgroup of order 5 and a unique subgroup of order 19.

   b) Show that G has a subgroup of index 3. [Hint: You may use the fact that if $H, K \leq G$ and $H \trianglelefteq G$ then $HK \leq G$.]