MATH 516 # Hour Exam **Radford** **10/13/06 10AM**

Name (print) __________________________________

(1) *Return* this exam copy with your exam booklet. (2) *Write* your solutions in your exam booklet. (3) *Show* your work. (4) There are *five questions* on this exam. (5) Each question counts 20 points. (6) *You are expected to abide by the University's rules concerning academic honesty.*

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1. Let $G = <a>$ be a cyclic group of order 22.

 (a) Find the number of subgroups of G.

 (b) Find $|a^{-94}|$.

 (c) List the generators of G in the form a^ℓ, where $0 \leq \ell < 22$.

 (d) List the elements of $<a^{46}>$ in the form a^ℓ, where $0 \leq \ell < 22$.

 (e) Draw the lattice diagram for G.

2. Let $\mathrm{GL}_2(\mathbf{R})$ be the group of invertible 2×2 matrices with real coefficients under matrix multiplication and set

$$G = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbf{R} \text{ and } ab \neq 0\}.$$

 (a) Show that $G \leq \mathrm{GL}_2(\mathbf{R})$.

 (b) The group G acts on $A = \mathbf{R}^2$ on the left by matrix multiplication. Describe the G-orbits of A. How many are there?

3. Let $f, g : G \longrightarrow G'$ be homomorphisms.

 (a) Show that

$$H = \{a \in G \mid f(a) = g(a)\}$$

 is a subgroup of G.

 (b) Let $a \in G$. Show, by induction, that $f(a^n) = f(a)^n$ for all $n \geq 0$. (Use the definition $a^0 = e$ and $a^{n+1} = aa^n$ for $n \geq 0$.)

4. Let G be a finite group and suppose that $H, K \leq G$.

 (a) Suppose that $H, K \trianglelefteq G$. Show that $HK \trianglelefteq G$.

 (b) Suppose that $|H| = \ell$ and H is the only subgroup of G of order ℓ. Show that $H \trianglelefteq G$.

5. Let G be a finite group of order $3{\cdot}5{\cdot}19$.

 (a) Show that G has a normal subgroup of order 19.

 (b) Show that G has a subgroup of index 3.