MATH 516 # Hour Exam Solution **Radford** **10/23/08**

Name (print) ____________________

(1) *Return* this exam copy with your exam booklet. (2) *Write* your solutions in your exam booklet. (3) *Show* your work. (4) There are *four questions* on this exam. (5) Each question counts 25 points. (6) *You are expected to abide by the University's rules concerning academic honesty.*

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1. (**25 points**) Let $G = <a>$ be a cyclic group of order 35.

(a) (**5 pts**) Find the number of subgroups of G.

Solution: The number of subgroups of G is the number if divisors of $|G| = 5{\cdot}7$; thus $\boxed{4}$.

(b) (**5 pts**) Find $|a^{-77}|$.

Solution: $|a^{-77}| = |<a^{-77}>| = 35/(-77, 35) = 5{\cdot}7/(-7{\cdot}11, 5{\cdot}7) = 5{\cdot}7/7 = \boxed{5}$.

(c) (**5 pts**) List the generators of G in the form a^ℓ, where $0 \leq \ell < 35$.

Solution: a^ℓ generates G if and only if $(\ell, 35) = 1$. Thus $0 \leq \ell < 35$ and multiples of $5, 7$ are excluded which means a^ℓ, where

$$\boxed{\ell \in \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 16, 17, 18, 19, 22, 23, 24, 26, 27, 29, 31, 32, 33\}}.$$

(d) (**5 pts**) List the elements of $<a^{205}>$ in the form a^ℓ, where $0 \leq \ell < 35$.

Solution: $<a^{205}> = <a^{(205,35)}> = <a^{(5\cdot 41, 5\cdot 7)}> = <a^5> = \boxed{\{e = a^0, a^5, a^{10}, a^{15}, a^{20}, a^{25}, a^{30}\}}$.

(e) (**5 pts**) Is G the only abelian group of order 35? Justify your answer.

Solution: There is only one possible primary decomposition of such a group, namely $\mathbf{Z}_5 \times \mathbf{Z}_7$ (up to isomophism). Thus $\boxed{\text{yes}}$.

2. (**25 points**) Let G be a group, let $H \leq G$, let A be the set of left cosets of H in G, and finally let $\pi : G \longrightarrow S_A$ be the permutation representation induced by the left action of G on A given by $g{\cdot}(aH) = gaH$ for all $g \in G$ and $aH \in A$.

(a) Show that $\operatorname{Ker} \pi$ is the largest normal subgroup of G which is contained in H.

Solution: Suppose $a \in \operatorname{Ker} \pi$. Then $\pi(a)(gH) = gH$, or equivalently $agH = gH$, for all $g \in G$. Letting $g = e$ note that $aH = H$ which implies $a \in H$. Therefore $\operatorname{Ker} \pi \subseteq H$. Kernels are always normal subgroups. (**5 pts**)

Conversely, suppose $N \trianglelefteq G$ and $N \subseteq H$. Let $a \in N$ and $g \in G$. Then $agH = g(g^{-1}ag)H = gH$ since $g^{-1}ag \in N \subseteq H$. Therefore $\pi(a)(gH) = gH$ for all $gH \in A$ which means $a \in \operatorname{Ker} \pi$; hence $N \subseteq \operatorname{Ker} \pi$. (**5 pts**)

(b) Now suppose that G is finite, $|G : H| = n$, and $|G| > n!$. Show that H contains a normal subgroup $(e) \neq N$ of G.

Solution: Note that $|A| = |G : H| = n$. If $|\mathrm{Ker}\,\pi| = 1$ then π is injective and therefore $n! < |G| = |\pi(G)| \leq |S_A| = n!$, a contradiction. Therefore $(e) \neq \mathrm{Ker}\,\pi \subseteq H$; the inclusion follows by part (a). (**15 pts**)

3. (**25 points**) Let $f, g : G \longrightarrow G'$ be group homomorphisms.

(a) (**5 pts**) Suppose that $S \subseteq G$ is a non-empty set. Show that $f(<S>) = <f(S)>$.

Solution: Perhaps the most easily seen way is to use the constructive formulation of $<S>$ as many did. Here is an element free proof.

$S \subseteq <S>$ implies $f(S) \subseteq <f(S)> \leq G'$ and thus $S \subseteq f^{-1}(<f(S)>) \leq G$. Therefore $<S> \subseteq f^{-1}(<f(S)>)$ and consequently $f(<S>) \subseteq <f(S)>$.

Conversely, $S \subseteq <S> \leq G$ implies $f(S) \subseteq f(<S>) \leq G'$. Thus $<f(S)> \subseteq f(<S>)$.

(b) (**6 pts**) Suppose that f is surjective. Use part (a) to show that if G is finitely generated (respectively cyclic) implies G' is finitely generated (respectively cyclic).

Solution: Suppose G is finitely generated. Then $G = <S>$ for some finite subset S of G. By part (a), $G' = f(G) = f(<S>) = <f(S)>$. Since $f(S)$ is finite, G' is finitely generated. When G is cyclic we can take $S = \{a\}$ for some $a \in G$ in which case $f(S) = \{f(a)\}$ generates G'; thus G' is cyclic.

(c) (**6 pts**) Show that $H = \{a \in G \,|\, f(a) = g(a)\} \leq G$.

Solution: $e \in H$ as $f(e) = e' = g(e)$. Therefore $H \neq \emptyset$. Let $a, b \in H$. Then

$$f(ab^{-1}) = f(a)f(b^{-1}) = f(a)f(b)^{-1} = g(a)g(b)^{-1} = g(a)g(b^{-1}) = g(ab^{-1})$$

which implies $ab^{-1} \in H$. Therefore $H \leq G$.

(d) (**8 pts**) Suppose $S \subseteq G$ generates G and $f(s) = g(s)$ for all $s \in S$. Show that $f = g$.

Solution: We use part (c). By assumption $S \subseteq H \leq G$. Therefore $G = <S> \subseteq H \subseteq G$ which implies $H = G$; this is equivalent to the conclusion.

4. (**25 points**) Let G be a finite group of order $5{\cdot}7{\cdot}17$.

(a) (**15 pts**) Show that G has a normal subgroup of order 7 or 17.

Solution: Let H_p denote a Sylow p-subgroup of G. By the Sylow Theorems $n_{17} = 1, 35(= 1+17{\cdot}2)$ and $n_7 = 1, 85(= 1+7{\cdot}12)$. If $n_{17} = 35$ and $n_7 = 85$ then the number of generators of the Sylow 17-subgroups plus the same of the Sylow 7-subgroups is $35{\cdot}16+85{\cdot}6 > 35{\cdot}17 = |G|$, contradiction. Therefore $n_{17} = 1$ or $n_7 = 1$ which means $H_{17} \trianglelefteq G$ or $H_7 \trianglelefteq G$.

(b) (**10 pts**) Show that G has a subgroup of index 5. [Hint: Consider the product of two appropriate Sylow p-subgroups.]

Solution: Since one of H_{17}, H_7 is a normal subgroup of G, $H_7H_{17} \leq G$. Now $H_7 \cap H_{17} \subseteq H_7, H_{17}$ means $|H_7 \cap H_{17}|$ divides $|H_7| = 7$ and $|H_{17}| = 17$. Therefore $|H_7 \cap H_{17}| = 1$ and $|H_7H_{17}| = |H_7||H_{17}|/|H_7 \cap H_{17}| = 7{\cdot}17/1 = 7{\cdot}17$. Since $|G : H_7H_{17}| = |G|/|H_7H_{17}| = 5{\cdot}7{\cdot}17/7{\cdot}17 = 5$ we are done.