1. (25 points) Let $G = \langle a \rangle$ be a cyclic group of order 35.
   (a) (5 pts) Find the number of subgroups of $G$.
   \textit{Solution}: The number of subgroups of $G$ is the number of divisors of $|G| = 5 \cdot 7$; thus $4$.
   (b) (5 pts) Find $|a^{-77}|$.
   \textit{Solution}: $|a^{-77}| = |\langle a^{-77} \rangle| = 35/(−77, 35) = 5 \cdot 7/(−7 \cdot 11, 5 \cdot 7) = 5 \cdot 7/7 = 5$.
   (c) (5 pts) List the generators of $G$ in the form $a^\ell$, where $0 \leq \ell < 35$.
   \textit{Solution}: $a^\ell$ generates $G$ if and only if $(\ell, 35) = 1$. Thus $0 \leq \ell < 35$ and multiples of $5, 7$ are excluded which means $a^\ell$, where
   \[ \ell \in \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 16, 17, 18, 19, 22, 23, 24, 26, 27, 29, 31, 32, 33\}. \]
   (d) (5 pts) List the elements of $\langle a^{205} \rangle$ in the form $a^\ell$, where $0 \leq \ell < 35$.
   \textit{Solution}: $\langle a^{205} \rangle = \langle a^{(205,35)} \rangle = \langle a^{(5\cdot41,5\cdot7)} \rangle = \langle a^5 \rangle = \{e, a^5, a^{10}, a^{15}, a^{20}, a^{25}, a^{30}\}$.
   (e) (5 pts) Is $G$ the only abelian group of order 35? Justify your answer.
   \textit{Solution}: There is only one possible primary decomposition of such a group, namely $\mathbb{Z}_5 \times \mathbb{Z}_7$ (up to isomorphism). Thus $\text{yes}$.

2. (25 points) Let $G$ be a group, let $H \leq G$, let $A$ be the set of left cosets of $H$ in $G$, and finally let $\pi: G \rightarrow S_A$ be the permutation representation induced by the left action of $G$ on $A$ given by $g \cdot (aH) = gaH$ for all $g \in G$ and $aH \in A$.
   (a) Show that Ker $\pi$ is the largest normal subgroup of $G$ which is contained in $H$.
   \textit{Solution}: Suppose $a \in \text{Ker } \pi$. Then $\pi(a)(gH) = gH$, or equivalently $agH = gH$, for all $g \in G$. Letting $g = e$ note that $aH = H$ which implies $a \in H$. Therefore Ker $\pi \subseteq H$. Kernels are always normal subgroups. (5 pts)
   Conversely, suppose $N \trianglelefteq G$ and $N \subseteq H$. Let $a \in N$ and $g \in G$. Then $agH = g(g^{-1}ag)H = gH$ since $g^{-1}ag \in N \subseteq H$. Therefore $\pi(a)(gH) = gH$ for all $gH \in A$ which means $a \in \text{Ker } \pi$; hence $N \subseteq \text{Ker } \pi$. (5 pts)
(b) Now suppose that $G$ is finite, $|G : H| = n$, and $|G| > n!$. Show that $H$ contains a normal subgroup $(e) \neq N$ of $G$.

**Solution:** Note that $|A| = |G : H| = n$. If $|\text{Ker } \pi| = 1$ then $\pi$ is injective and therefore $n! < |G| = |\pi(G)| \leq |S_A| = n!$, a contradiction. Therefore $(e) \neq \text{Ker } \pi \subseteq H$; the inclusion follows by part (a). (15 pts)

3. (25 points) Let $f, g : G \rightarrow G'$ be group homomorphisms.

(a) (5 pts) Suppose that $S \subseteq G$ is a non-empty set. Show that $f(<S>) = <f(S)>$.

**Solution:** Perhaps the most easily seen way is to use the constructive formulation of $<S>$ as many did. Here is an element free proof.

$S \subseteq <S>$ implies $f(S) \subseteq <f(S)> \subseteq G'$ and thus $S \subseteq f^{-1}(<f(S)>) \leq G$. Therefore $<S> \subseteq f^{-1}(<f(S)>)$ and consequently $f(<S>) \subseteq <f(S)>$.

Conversely, $S \subseteq <S> \leq G$ implies $f(S) \subseteq f(<S>) \leq G'$. Thus $<f(S) > \subseteq f(<S>)$.

(b) (6 pts) Suppose that $f$ is surjective. Use part (a) to show that if $G$ is finitely generated (respectively cyclic) implies $G'$ is finitely generated (respectively cyclic).

**Solution:** Suppose $G$ is finitely generated. Then $G = <S>$ for some finite subset $S$ of $G$. By part (a), $G' = f(G) = f(<S>) = <f(S)>$. Since $f(S)$ is finite, $G'$ is finitely generated. When $G$ is cyclic we can take $S = \{a\}$ for some $a \in G$ in which case $f(S) = \{f(a)\}$ generates $G'$; thus $G'$ is cyclic.

(c) (6 pts) Show that $H = \{a \in G | f(a) = g(a)\} \leq G$.

**Solution:** $e \in H$ as $f(e) = e' = g(e)$. Therefore $H \neq \emptyset$. Let $a, b \in H$. Then

$$f(ab^{-1}) = f(a)f(b^{-1}) = f(a)f(b)^{-1} = g(a)g(b)^{-1} = g(a)g(b^{-1}) = g(ab^{-1})$$

which implies $ab^{-1} \in H$. Therefore $H \leq G$.

(d) (8 pts) Suppose $S \subseteq G$ generates $G$ and $f(s) = g(s)$ for all $s \in S$. Show that $f = g$.

**Solution:** We use part (c). By assumption $S \subseteq H \leq G$. Therefore $G = <S> \subseteq H \subseteq G$ which implies $H = G$; this is equivalent to the conclusion.

4. (25 points) Let $G$ be a finite group of order 5-17.

(a) (15 pts) Show that $G$ has a normal subgroup of order 7 or 17.

**Solution:** Let $H_p$ denote a Sylow $p$-subgroup of $G$. By the Sylow Theorems, $n_{17} = 1, 35 (= 1 + 17 \cdot 2)$ and $n_7 = 1, 85 (= 1 + 7 \cdot 12)$. If $n_{17} = 35$ and $n_7 = 85$ then the number of generators of the Sylow 17-subgroups plus the same of the Sylow 7-subgroups is $35 \cdot 16 + 85 \cdot 6 > 35 \cdot 17 = |G|$, contradiction. Therefore $n_{17} = 1$ or $n_7 = 1$ which means $H_{17} \leq G$ or $H_7 \leq G$.

(b) (10 pts) Show that $G$ has a subgroup of index 5. [Hint: Consider the product of two appropriate Sylow $p$-subgroups.]

**Solution:** Since one of $H_{17}, H_7$ is a normal subgroup of $G$, $H_{17}H_7 \leq G$. Now $H_7 \cap H_{17} \leq H_7, H_{17}$ means $|H_7 \cap H_{17}|$ divides $|H_7| = 7$ and $|H_{17}| = 17$. Therefore $|H_7 \cap H_{17}| = 1$ and $|H_7H_{17}| = |H_7||H_{17}|/|H_7 \cap H_{17}| = 7 \cdot 17 / 1 = 7 \cdot 17$. Since $|G : H_7H_{17}| = |G|/|H_7H_{17}| = 5 \cdot 17 / 7 \cdot 17 = 5$ we are done.