1. Let $G$ be an additive group and $S$ be a non-empty set. Then $G = \text{Fun}(S, G)$, the set of all functions $f : S \rightarrow G$, is an additive group, where $(f + g)(s) = f(s) + g(s)$ for all $f, g \in G$ and $s \in S$. Let $A(S, G)$ be the subset of $G$ consisting of all $f \in G$ which satisfy $f(s) = 0$ for all but finitely many $s \in S$.

(a) Show that $A(S, G) \leq G$.

(b) Let $\iota : S \rightarrow A(S, \mathbb{Z})$ be the function defined by $\iota(s)(s') = \begin{cases} 1 & s' = s; \\ 0 & s' \neq s. \end{cases}$ Show that $(\iota, A(S, \mathbb{Z}))$ is a free abelian group on $S$.

2. Let $R^\times$ be the group of units of $R$.

(a) Suppose that $a \in R$ and $\{1, a, a^2, a^3, \ldots\}$ is a finite set. Show that $a \in R^\times$ or $ab = 0 = ba$ for some non-zero $b \in R$.

(b) Suppose that $R$ is finite. Show that any element of $R$ is a unit or is a zero divisor.

(c) Let $R = M_n(F)$, where $n \geq 1$, and let $a \in R$. Show that $a \in R^\times$ or $ab = 0 = ba$ for some non-zero $b \in R$. [Hint: Is $\{1, a, a^2, a^3, \ldots\}$ linearly independent?]

3. You may assume that if $a, b \in R$ commute then the binomial theorem holds for them; that is $(a + b)^n = \sum_{\ell=0}^n \binom{n}{\ell} a^{n-\ell}b^\ell$ for all $n \geq 0$. Also, you may assume the exponent laws.

(a) Suppose $a, b \in R$ are nilpotent and $ab = ba$. Show that $a \pm b$ is nilpotent.

(b) Suppose $a, r \in R$, where $a$ is nilpotent and $ar = ra$. Show that $ar$ is nilpotent.

(c) Suppose that $R$ is commutative and $N$ is the set of nilpotent elements of $R$. Show that $N$ is an ideal of $R$.

(d) Suppose $R = M_2(F)$. Find nilpotent $a, b \in R$ such that $a + b$ and $ab$ are not nilpotent. Justify your answer.
4. Let \( F((x)) = \{ \sum_{n \geq N} a_n x^n \mid N \in \mathbb{Z}, a_n \in F \ \forall \ n \geq N \} \) be the (commutative) ring of formal Laurent series with coefficients in \( F \). Writing elements of \( F((x)) \) as \( \sum a_n x^n \) we have
\[
\sum a_n x^n + \sum b_n x^n = \sum (a_n + b_n) x^n \quad \text{and} \quad \left( \sum a_n x^n \right) \left( \sum b_n x^n \right) = \sum c_n x^n,
\]
where \( c_n = \sum_{i+j=n} a_i b_j \) for all \( n \in \mathbb{Z} \). Observe that \( F[[x]] \) is a subring of \( F((x)) \).

(a) Show that a non-zero element of \( F((x)) \) has a multiplicative inverse.

(b) Let \( 0 \neq f(x) = \sum_{n=0}^{\infty} a_n x^n \in F[[x]] \). Show that \( f(x)^{-1} \in F[[x]] \) if and only if \( a_0 \neq 0 \).

5. Suppose that \( R \) is commutative and for all \( a \in R \) there is a positive integer \( n > 1 \) such that \( a^n = a \). Show that every prime ideal of \( R \) is maximal.