1. Let $R$ be a non-zero Boolean ring with identity (a ring with identity such that $a^2 = a$ for all $a \in R$).

(a) Show that $R$ is commutative.

(b) Suppose that $e \in R$. Show that $Re$ and $R(1 - e)$ are ideals of $R$ and $R = Re \oplus R(1 - e)$.

(c) Suppose that $R$ is finite. Show that $R \simeq \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

2. Suppose that $R$ is any ring.

(a) Show that the direct product of abelian groups $R = \mathbb{Z} \times R$ is a ring with identity, where the product is defined by $(m, a) \cdot (n, b) = (mn, n \cdot a + m \cdot b + ab)$ for all $(m, a), (n, b) \in R$, and $\iota : R \to R$ defined by $\iota(a) = (0, a)$ for all $a \in R$ is an injective ring homomorphism.

(b) Show that there is an abelian group $A$ and an injective ring homomorphism $\gamma : R \to \text{End}(A)$.

Remark: Part (b) may be thought of as Cayley’s Theorem for rings.

3. Find all irreducible polynomials in $\mathbb{Z}_2[x]$ of degrees 2, 3, or 4. [Hint: Which ones are reducible?]

4. Let $R$ be a ring, let $I$ be a non-empty set, let $\{M_i\}_{i \in I}$ be an indexed family of left $R$-modules, and let $\prod_{i \in I} M_i = \{ f : I \to \bigcup_{i \in I} M_i \mid f(i) \in M_i \forall i \in I \}$.

(a) Show that $\prod_{i \in I} M_i$ is a left $R$-module, where

$$(f + g)(i) = f(i) + g(i) \quad \text{and} \quad (r \cdot f)(i) = r \cdot (f(i))$$

for all $f, g \in \prod_{i \in I} M_i$, $r \in R$, and $i \in I$.

(b) For $j \in I$ define $\pi_j : \prod_{i \in I} M_i \to M_j$ by $\pi_j(f) = f(j)$ for all $f \in \prod_{i \in I} M_i$. Show that $(\{\pi_i\}_{i \in I}, \prod_{i \in I} M_i)$ is a product of $\{M_i\}_{i \in I}$.

5. We continue with Exercise 4. For $i_0 \in I$ define $j_{i_0} : M_{i_0} \to \prod_{i \in I} M_i$ by $j_{i_0}(m)(i) = \begin{cases} m : i = i_0 \\ 0 : i \neq i_0 \end{cases}$. Show that $(\{j_i\}_{i \in I}, M)$ is a direct sum of $\{M_i\}_{i \in I}$ for some submodule $M$ of $\prod_{i \in I} M_i$. 