Math 517 **Spring 2007** **Radford**

Written Homework # 3 Solution

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1. (**25 points**) (1) First of all we show that $\varphi : R/Ra \times R/Rb \longrightarrow R/Rc$ given by $(r+Ra, s+Rb) \mapsto rs+Rc$ is a well-defined function. Suppose that $r+Ra = r'+Ra$ and $s+Rb = s'+Rb$, were $r, r', s, s' \in R$. As c divides a, b we have $Ra, Rb \subseteq Rc$. Since $r'-r \in Ra \subseteq Rc$ and $s'-s \in Rb \subseteq Rc$ for some $x, y \in R$, $r'-r = xc$ and $s'-s = yc$. Therefore $r's'-rs = (r+xc)(s+yc)-rs = ryc+xc(s+yc) \in Rc$ which means $r's'+Rc = rs+Rc$. We have shown that φ is a well-defined function. The reader is left with the direct calculation that φ is R-bilinear; that is

$$\varphi((r+Ra)+(r'+Ra), s+Rb) = \varphi(r+Ra, s+Rb) + \varphi(r'+Ra, s+Rb),$$

$$\varphi(r+Ra, (s+Rb)+(s'+Rb)) = \varphi(r+Ra, s+Rb) + \varphi(r+Ra, s'+Rb),$$

and

$$\varphi(r'\cdot(r+Ra), s+Rb) = \varphi(r+Ra, r'\cdot(s+Rb))$$

for all $r, r', s, s' \in R$.

By the universal mapping property of the tensor product of R-modules over a commutative ring there is homomorphism of R-modules

$$f : R/Ra \otimes_R R/Rb \longrightarrow R/Rc$$

such that $f \circ \imath = \varphi$, where $\imath : R/Ra \times R/Rb \longrightarrow R/Ra \otimes_R R/Rb$ is defined by $\imath(r+Ra, s+Rb) = (r+Ra)\otimes(s+Rb)$. Therefore $f((r+Ra)\otimes(s+Rb)) = rs+Rc$. (**10**)

(2) We show that g is well defined first of all. Suppose that $r+Rc = r'+Rc$. Then $r'-r = zc$ for some $z \in R$. As $c = xa+yb$ we have

$r' = r + zc = r + zxa + zyb$ and therefore

$$\begin{array}{ll} (r' + Ra)\otimes(1 + Rb) & \\ \quad = & (r + zxa + zyb + Ra)\otimes(1 + Rb) \\ \quad = & (r + zyb + Ra)\otimes(1 + Rb) \\ \quad = & ((r + Ra) + (zyb + Ra))\otimes(1 + Rb) \\ \quad = & (r + Ra)\otimes(1 + Rb) + (zyb + Ra)\otimes(1 + Rb) \\ \quad = & (r + Ra)\otimes(1 + Rb) + b\cdot(zy + Ra)\otimes(1 + Rb) \\ \quad = & (r + Ra)\otimes(1 + Rb) + (zy + Ra)\otimes b\cdot(1 + Rb) \\ \quad = & (r + Ra)\otimes(1 + Rb) + (zy + Ra)\otimes(b + Rb) \\ \quad = & (r + Ra)\otimes(1 + Rb) + (zy + Ra)\otimes(0 + Rb) \\ \quad = & (r + Ra)\otimes(1 + Rb). \end{array}$$

Thus g is well-defined. Since

$$\begin{aligned} g((r + Rc) + r'\cdot(r'' + Rc)) &= g((r + r'r'') + Rc) \\ &= ((r + r'r'') + Ra)\otimes(1 + Rb) \\ &= ((r + Ra) + (r'r'' + Ra))\otimes(1 + Rb) \\ &= (r + Ra)\otimes(1 + Rb) + (r'r'' + Ra)\otimes(1 + Rb) \\ &= (r + Ra)\otimes(1 + Rb) + r'\cdot((r'' + Ra)\otimes(1 + Rb)) \\ &= g(r + Rc) + r'\cdot g(r'' + Rc) \end{aligned}$$

it follows that g is a homomorphism of left R-modules.

We show that the module maps f and g are inverses. To this end we need only check that

$$\begin{aligned} (g\circ f)((r + Ra)\otimes(s + Rb)) &= g(f((r + Ra)\otimes(s + Rb)) \\ &= g(rs + Rc) \\ &= (rs + Ra)\otimes(1 + Rb) \\ &= (s\cdot(r + Ra))\otimes(1 + Rb) \\ &= (r + Ra)\otimes(s\cdot(1 + Rb)) \\ &= (r + Ra)\otimes(s + Rb) \end{aligned}$$

and

$$(f\circ g)((r + Rc)) = f(g((r + Ra)) = f((r + Ra)\otimes(1 + Rb)) = r1 + Rc = r + Rc$$

for all $r, s \in R$. (**10**)

(3) The hypothesis of (2) is met in this case. (**5**)

2. (**20 points**) (1) We may assume that D is a subring of F. Suppose that F is a submodule of a free left D-module M and let $\{m_i\}_{i\in I}$ be a basis for M. Let $a, b \in D\backslash 0$ and write

$$\frac{1}{b} = a_1{\cdot}m_{i_1} + \cdots + a_s{\cdot}m_{i_s},$$

where $i_1, \ldots i_s \in I$ are distinct and $a_1, \ldots, a_s \in D\backslash 0$. Then

$$\frac{a}{b} = aa_1{\cdot}m_{i_1} + \cdots + aa_s{\cdot}m_{i_s}$$

and

$$1 = ba_1{\cdot}m_{i_1} + \cdots + ba_s{\cdot}m_{i_s}.$$

Since D is an integral domain none of the coefficients in the two preceding equations are zero. Therefore $\mathrm{supp}(1) = \{m_{i_1}, \ldots, m_{i_s}\} = \mathrm{supp}(r)$ for all $r \in F\backslash 0$. Writing

$$\frac{1}{ba_1} = c_1{\cdot}m_{i_1} + \cdots + c_s{\cdot}m_{i_s}$$

for some $c_1, \ldots, c_s \in D$ we have that

$$1 = ba_1c_1{\cdot}m_{i_1} + \cdots + ba_1c_s{\cdot}m_{i_s}$$

from which we deduce that $ba_1 = ba_1c_1$ and therefore $c_1 = 1$.

The composition of the projection $D{\cdot}m_{i_1} \oplus \cdots \oplus D{\cdot}m_{i_s} \longrightarrow D{\cdot}m_{i_1}$ to the first summand followed by the isomorphism $D{\cdot}m_{i_1} \simeq D$ $(d{\cdot}m_{i_1} \mapsto d)$ restricts to an injective homomorphism of left R-modules $f : F \longrightarrow$D. Since $f(\dfrac{1}{ba_1}) = 1$ it follows that f is surjective. Therefore f is an isomorphism of left D-modules which means that F is a free left D-module. By part (3) of WH2 it follows that $F = D$. (**15**)

(2) If $F = D$ then it is a free, hence a projective, D-module. Conversely, suppose that F is projective. Then it is isomorphic to a submodule of a free D-module. Without loss of generality we may assume that F is a submodule of a free D-module. Therefore $F = D$ by part (1). (**5**)

3. (**35 points**) (1) Let $m_1', m_2' \in M'$ and $m_1'', m_2'' \in M''$. First of all we show that $f' + f'' : M' + M'' \longrightarrow Q$ is well-defined. Suppose that $m_1' + m_1'' = m_2' + m_2''$. Then $m_1' - m_2' = m_2'' - m_1'' \in M' \cap M''$ which means

$$f'(m_1') - f'(m_2') = f'(m_1' - m_2') = f''(m_2'' - m_1'') = f''(m_2'') - f''(m_1'')$$

and therefore

$$f'(m_1') + f''(m_1'') = f'(m_2') + f''(m_2'').$$

That $f' + f''$ is a module map follows by

$$\begin{array}{rl} & (f' + f'')((m_1' + m_1'') + (m_2' + m_2'')) \\ = & (f' + f'')((m_1' + m_2') + (m_1'' + m_2'')) \\ = & f'((m_1' + m_2') + f''(m_1'' + m_2'') \\ = & f'(m_1') + f'(m_2') + f''(m_1'') + f''(m_2'') \\ = & f'(m_1') + f''(m_1'') + f'(m_2') + f''(m_2'') \\ = & (f' + f'')(m_1' + m_1'') + (f' + f'')(m_2' + m_2'') \end{array}$$

and

$$\begin{array}{rcl} (f' + f'')(r{\cdot}(m_1' + m_1'')) & = & (f' + f'')(r{\cdot}m_1' + r{\cdot}m_1'') \\ & = & f'(r{\cdot}m_1') + f''(r{\cdot}m_1'') \\ & = & r{\cdot}f'(m_1') + r{\cdot}f''(m_1'') \\ & = & r{\cdot}(f'(m_1') + f''(m_1'')) \\ & = & r{\cdot}((f' + f'')(m_1' + m_1'')) \end{array}$$

for all $r \in R$. (**10**)

To complete the proof that $(M' + M'', f' + f'') \in \mathcal{S}$ we need to show that $(M_0, f_0) \leq (M' + M'', f' + f'')$. Since $M_0 \subseteq M', M''$ and $f'|_{M_0} = f_0$, $f''|_{M_0} = f_0$, we see that $M_0 \subseteq M' + M''$ and for $m \in M_0$ ($\subseteq M'$) that

$$(f' + f'')(m) = (f' + f'')(m + 0) = f'(m) + f''(0) = f_0(m).$$

Thus $(f' + f'')|_{M_0} = f_0$ and hence $(M_0, f_0) \leq (M' + M'', f' + f'')$. Therefore $(M' + M'', f' + f'') \in \mathcal{S}$. **Note**: By the same argument $(M', f'), (M'', f'') \leq (M' + M'', f' + f'')$. (**5**)

(2) First of all $\mathcal{S}$ is a partially ordered set; that is:

(PO.1) $(f', M') \leq (f', M')$ for all $(f', M') \in \mathcal{S}$;

(PO.2) If $(f', M'), (f'', M'') \in \mathcal{S}$ satisfy $(f', M') \leq (f'', M'')$ and $(f'', M'') \leq (f', M')$ then $(f', M') = (f'', M'')$;

(PO.3) If $(f', M'), (f'', M''), (f''', M''') \in \mathcal{S}$ satisfy $(f', M') \leq (f'', M'')$ and $(f'', M'') \leq (f''', M''')$ then $(f', M') \leq (f''', M''')$.

To see (PO.1) note that $M' \subseteq M'$ and $f'|_{M'} = f'$ for $(f', M') \in \mathcal{S}$. The hypothesis of (PO.2) implies that $M' \subseteq M'' \subseteq M'$, hence $M' \subseteq M''$ and thus $f'' = f''|_{M''} = f''|_{M'} = f'$. Therefore $(M', f') = (M'', f'')$. As for (PO.3), $(f', M') \leq (f'', M'') \leq (f''', M''')$ implies $M' \subseteq M'' \subseteq M''$; thus $M' \subseteq M'''$ and $f'''|_{M'} = (f'''|_{M''})|_{M'} = f''|_{M'} = f'$. Therefore $(M', f') \leq (M''', f''')$. (**5**)

Let $\mathcal{C}$ be a chain in $\mathcal{S}$; that is a non-empty subset of $\mathcal{S}$ such that for all $i, j \in I$ either $(M'_i, f'_i) \leq (M'_j, f'_j)$ or $(M'_j, f'_j) \leq (M'_i, f'_i)$. Then $N = \cup_{i \in I} M'_i$ is a submodule of M. To see this, first of all note that $N \neq \emptyset$ since $\mathcal{C} \neq \emptyset$. Let $n, n' \in N$ and $r \in R$. Then $n \in M_i$ and $n' \in M_{i'}$ for some $i, i' \in I$. Since $\mathcal{C}$ is a chain either $(M'_i, f'_i) \leq (M'_{i'}, f'_{i'})$ or $(M'_{i'}, f_{i'}) \leq (M'_i, f'_i)$. Thus $M_i \subseteq M_{i'}$ or $M_{i'} \subseteq M_i$. Without loss of generality we may assume $M_i \subseteq M_{i'}$. Thus $n, n' \in M_{i'}$ which means $n + r{\cdot}n' \in M_{i'} \subseteq N$. Therefore N is a submodule of M.

Then there is a module map $f : N \longrightarrow Q$ described as follows. Let $n \in N$. Then $n \in M'_i$ for some $i \in I$. Set $f(n) = f'_i(n)$.

Suppose that there is a such a module map. Then $(N, f) \in \mathcal{S}$ and $(M'_i, f'_i) \leq (N, f)$ for all $i \in I$. We have noted that $(M_0, f_0) \in \mathcal{S}$. Let $(M'_i, f'_i) \in \mathcal{C}$. Then $(M_0, f_0) \leq (M'_i, f'_i)$ which implies $M_0 \subseteq M'_i \subseteq N$ and $f|_{M_0} = f|0$, $f|_{M_i} = f_i$ by our construction of f. Therefore $(M_0, f_0) \leq (M'_i, f'_i) \leq (N, f)$ which shows that $(N, f) \in \mathcal{S}$ and that (N, f) is an upper bound for $\mathcal{C}$. It remains to show that f does in fact exist. (**10**)

First of all f is *well-defined*. Suppose that $n \in N$. Then $n \in M'_i$ for some $i \in I$, where $(M'_i, f'_i) \in \mathcal{C}$. Suppose that $n \in M'_{i'}$, where $(M'_{i'}, f'_{i'}) \in \mathcal{C}$ also. Since $(M_i, f'_i) \leq (M'_{i'}, f'_{i'})$ or vice versa, we may assume $(M_i, f'_i) \leq (M'_{i'}, f'_{i'})$. Therefore $M_i \subseteq M_{i'}$ which means that $f'_{i'}(n) = f'_{i'}|_{M'_i}(n) = f'_i(n)$. Thus f is well defined. To see that f is a module map, let $n, n' \in N$. We have seen that $n, n' \in M_i$ for some $i \in I$. Since $f|_{M_i} = f'_i$ is a module map necessarily f is a module map. (**5**)

4. (**20 points**) (1) $0 \in L$ since $0{\cdot}m = 0 \in M'$. Suppose that $r, r'' \in L$ and

$r' \in R$. Then $(r + r'r'')\cdot m = r\cdot m + r'\cdot(r''\cdot m) \in M' + r'\cdot M' \subseteq M'$. Therefore L is a left ideal of R. (**5**)

(2) Let $r, r'' \in L$ and $r' \in R$. Then, using part (1), we see that $r\cdot m$, $r'r''\cdot m \in M'$. Thus

$$\begin{aligned} F(r + r'r'') &= f'((r + r'r'')\cdot m) \\ &= f'(r\cdot m + r'r''\cdot m) \\ &= f'(r\cdot m) + f'(r'r''\cdot m) \\ &= r\cdot f'(m) + r'r''\cdot f'(m) \\ &= r\cdot f'(m) + r'\cdot(r''\cdot f'(m)) \\ &= F(r) + r'\cdot F(r'') \end{aligned}$$

which shows that F is a module homomorphism. (**5**)

(3) g is *well-defined*. Suppose that $r, r' \in R$ and $r\cdot m = r'\cdot m$. Then $(r - r')\cdot m = 0$ which means that $r - r' \in L$. Therefore

$$G(r - r') = F(r - r') = f'((r - r')\cdot m) = f'(0) = 0$$

which means that $G(r) = G(r')$. Therefore $g(r\cdot m) = g(r'\cdot m)$.

Let $r, r'r'' \in R$. Then

$$\begin{aligned} g(r\cdot m + r'\cdot(r''\cdot m)) &= g((r + r'r'')\cdot m) \\ &= G(r + r'r'') \\ &= G(r) + G(r'r'') \\ &= G(r) + r'\cdot G(r'') \\ &= g(r\cdot m) + r'\cdot g(r''\cdot m) \end{aligned}$$

which shows that g is a module homomorphism.

Let $x \in M' \cap R\cdot m$. Then $x = r\cdot m$ for some $r \in R$ since $x \in R\cdot m$. Since $x \in M'$, $r \in L$. Therefore $g(x) = g(r\cdot m) = G(r) = F(r) = f'(r\cdot m) = f'(x)$. (**5**)

(4) Let $m \in M$. With $f' = f_e$ by parts (2) and (3) there is a homomorphism of left R-modules $g : R\cdot m \longrightarrow Q$ such that $g|_{M_e \cap R\cdot m} = f'|_{M_e \cap R\cdot m}$. By part (1) of Problem 3, $(M_e, f') \leq (f' + g, M_e + R\cdot m)$. By (PO.3) we conclude that $(f' + g, M_e + R\cdot m) \in \mathcal{S}$. Therefore $(M_e, f') = (f' + g, M_e + R\cdot m)$ which means $M_e = M_e + R\cdot m$. We have shown $m \in M_e$ and thus $M = M_e$. (**5**)