If $F$ and $K$ are fields then $F \subseteq K$ means that $F$ is a subfield of $K$. Also $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of rational, real, and complex numbers respectively. You must justify your answers.

1. Find:

(1) The degree of $\sqrt[3]{34}$ over $\mathbb{Q}$;

(2) $m_{\mathbb{Q}[\sqrt[3]{21}], \psi_{\sqrt[3]{21}}}(x)$;

(3) $m_{\mathbb{Q}[\sqrt[3]{21}, \sqrt[3]{21}]}(x)$.

2. Let $a = \sqrt[3]{n} \in \mathbb{R}$, where $n \geq 2$ and is an integer such that $p$ divides $n$ but $p^2$ does not divide $n$ for some prime $p$. Let $K = \mathbb{Q}[a]$.

(1) Determine $[K : \mathbb{Q}]$ and find a basis for $K$ as a vector space over $\mathbb{Q}$.

(2) Let $b = r + sa$, where $r, s \in \mathbb{Q}$ and $s \neq 0$. Show $b \notin \mathbb{Q}$ and find $m_{\mathbb{Q}, b}(x)$.

3. Let $a = \sqrt{1 + \sqrt{2}} \in \mathbb{R}$.

(1) Show that $a \notin \mathbb{Q}[\sqrt{2}]$.

(2) Find $m_{\mathbb{Q}, a}(x)$ and find $m_{\mathbb{Q}[\sqrt{2}], a}(x)$.

(3) Write $f(x) = m_{\mathbb{Q}, a}(x)$ as a product of linear factors over $\mathbb{C}$, determine a splitting field $K \subseteq \mathbb{C}$ for $f(x)$ over $\mathbb{Q}$, and determine $[K : \mathbb{Q}]$.

4. Let $F$ and $K$ be fields and $F \subseteq K$.

(1) Show that all $a \in K \setminus K_{alg}$ are transcendental over $K_{alg}$. (Thus $a$ is transcendental over $F$.)

(2) Suppose that $a \in K$ is transcendental over $F$. For $n \geq 1$ show that $a^n$ is transcendental over $F$ and that $[F(a) : F(a^n)] = n$. (Thus $F(a)$ is an algebraic extension of $F(a^n)$ of degree $n$.)