Math 517 **Spring 2007** **Radford**

Written Homework # 1

Due at the beginning of class 02/02/07 [1]

Throughout R, S are rings with unity and modules are unital.

1. Let I be a non-empty set and let $\{P_i\}_{i\in I}$ be an indexed family of left R-modules. A *product of the family* is a pair $(\{\pi_i\}_{i\in I}, P)$, where

(P.1) P is a left R-module and $\pi_i : P \longrightarrow P_i$ is a homomorphism of left R-modules for all $i \in I$, and

(P.2) If $(\{\pi'_i\}_{i\in I}, P')$ is a pair which satisfies (P.1) then there is a unique R-module homomorphism $\Phi : P' \longrightarrow P$ which satisfies $\pi_i \circ \Phi = \pi'_i$ for all $i \in I$.

Prove the following theorem:

Theorem 1 *Let R be a ring with unity, let I be a non-empty set, and let $\{P_i\}_{i\in I}$ be an indexed family of left R-modules.*

(1) *There is a product of the family $\{P_i\}_{i\in I}$.*

(2) *Suppose that $(\{\pi_i\}_{i\in I}, P)$ and $(\{\pi'_i\}_{i\in I}, P')$ are products of the family $\{P_i\}_{i\in I}$. Then there is a unique isomorphism of left R-modules $\Phi : P' \longrightarrow P$ which satisfies $\pi_i \circ \Phi = \pi'_i$ for all $i \in I$.*

[Hint: Let P be the set of all functions $f : I \longrightarrow \bigcup_{i\in I} P_i$ which satisfy $f(i) \in P_i$ for all $i \in I$. Show that P is a left R-module under the operations

$$(f + g)(i) = f(i) + g(i)$$

[1] Slightly revised 01/24/07.

and

$$(r \cdot f)(i) = r \cdot (f(i))$$

for all $f, g \in P$ and $i \in I$. Consider $\pi_i : P \longrightarrow P_i$ defined by $\pi_i(f) = f(i)$ for all $f \in P$ and $i \in I$.

2. Let I be a non-empty set. A *free R-module on I* is a pair $(\imath, F)$, where

(F.1) F is a left R-module and $\imath : I \longrightarrow F$ is a set map, and

(F.2) if $(\imath', F')$ is a pair which satisfies (F.1) then there is a unique R-module homomorphism $\Phi : F \longrightarrow F'$ which satisfies $\Phi \circ \imath = \imath'$.

Prove the following theorem:

Theorem 2 *Let R be a ring with unity and let I be a non-empty set.*

(1) *There is a free left R-module $(\imath, F)$ on I.*

(2) *Suppose that $(\imath, F)$ and $(\imath', F')$ are free left R-modules on I. Then there is a unique isomorphism of left R-modules $\Phi : F \longrightarrow F'$ which satisfies $\Phi \circ \imath = \imath'$.*

Suppose that $(\imath, F)$ is a free left R-module.

(3) $\mathrm{Im}\, \imath$ *generates F as a left R-module.*

(4) *$\imath$ is injective and $\{\imath(\ell)\}_{\ell \in I}$ is a basis for F.*

[Hint: For part (1), let F be the subset of the product P of the family $\{R_i\}_{i \in I}$, where $R_i = R$ for all $i \in I$, of Exercise 1 consisting of all functions with finite (which includes empty) support. For $f \in P$ the support of f is defined by

$$\mathrm{supp}\, f = \{i \in I \mid f(i) \neq 0\}.$$

]

3. Suppose that $f : R \longrightarrow S$ is a function and for $r \in R$ and $s \in S$ define $r \cdot s = f(r)s$.

(a) Show that f is a homomorphism of rings with unity and $\mathrm{Im}\, f$ is in the center of S if and only if S is a left R-module and

$$r \cdot (ss') = (r \cdot s)s' = s(r \cdot s') \tag{1}$$

for all $r \in R$ and $s, s' \in S$.

(b) Suppose that S has a left R-module structure $(S, \bullet)$ which satisfies (1). Define $F : R \longrightarrow S$ by $F(r) = r \bullet 1$ for all $r \in R$. Show that F is a homomorphism of rings with unity and $\operatorname{Im} F$ is in the center of S.

The ring S is called an *R-algebra* if ${}_RS$ and (1) is satisfied. The exercise shows there are two ways of describing an R-algebra.

4. Let $\mathbf{Z}$ be the ring of integers and $\mathbf{Q}$ be the field of rational numbers.

(a) Let $\imath : 2\mathbf{Z} \longrightarrow \mathbf{Z}$ be the inclusion. Show that $\imath \otimes \mathrm{Id} : 2\mathbf{Z} \otimes_{\mathbf{Z}} (\mathbf{Z}/2\mathbf{Z}) \longrightarrow \mathbf{Z} \otimes_{\mathbf{Z}} (\mathbf{Z}/2\mathbf{Z})$ is not injective.

(b) Show that $\mathbf{Q} \otimes_{\mathbf{Z}} A = (0)$ for all finite abelian groups A.

(c) Suppose that $f : M_R \longrightarrow M'_R$ and $g : {}_RN \longrightarrow {}_RN'$ are surjective maps of R-modules. Show that the homomorphism of abelian groups $f \otimes g : M \otimes_R N \longrightarrow M' \otimes_R N'$ is a surjective.