MATH 531 Written Homework 5 Radford DUE WEDNESDAY 12/05/07

We follow the notation of the text and that used in class. You may use results from the course materials on the class homepage and the text. **This version replaces the previous one**.

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1. Let F be an algebraically closed field of characteristic zero. We continue our study of $L = sl(n, F)$ using the notation of Exercise Set 4, the results of which you may use. We set $t_{k,\ell} = t_{\alpha_{k,\ell}}$ for all $\alpha_{k,\ell} \in \mathbf{\Phi}$ and set $\alpha_i = \alpha_{i,i+1}$ for all $1 \leq i \leq n-1$. The results of §8.5 of the text, as treated in §2.5.5 of the Chapter Notes, is the background material for this exercise.

(a) Show that $t_{k,\ell} = \dfrac{1}{2n}\,(e_{k\,k} - e_{\ell\,\ell})$ for all $\alpha_{k,\ell} \in \mathbf{\Phi}$. [Hint: Recall that $t_\alpha \in H$ is defined by $\kappa(t_\alpha, h) = \alpha(h)$ for all $\alpha \in \mathbf{\Phi}$ and $h \in H$.]

(b) Show that $(\alpha_{k,\ell},\ \alpha_{r,s}) = \dfrac{1}{2n}\,(\delta_{k,r} + \delta_{\ell,s} - \delta_{k,s} - \delta_{\ell,r})$ for all $\alpha_{k,\ell}, \alpha_{r,s} \in \mathbf{\Phi}$. [Hint: Recall that $(\alpha, \beta) = \kappa(t_\alpha, t_\beta) = \alpha(t_\beta)$ for all $\alpha, \beta \in \mathbf{\Phi}$.]

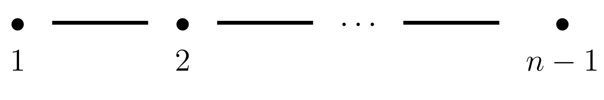
(c) Show that $||\alpha_{k,\ell}|| = \dfrac{1}{\sqrt{n}}$ for all $\alpha_{k,\ell} \in \mathbf{\Phi}$ and that the cosine of the angle θ between $\alpha_{k,\ell}, \alpha_{r,s} \in \mathbf{\Phi}$ is $\dfrac{1}{2}\,(\delta_{k,r} + \delta_{\ell,s} - \delta_{k,s} - \delta_{\ell,r})$. (Thus θ is one of $0, \pi/3, \pi/2, 2\pi/3$, or π.)

(d) Show that $\Delta = \{\alpha_1, \ldots, \alpha_{n-1}\}$ is a base for $\mathbf{\Phi}$, meaning that Δ is a basis for H^* and that all $\beta \in \mathbf{\Phi}$ can be written $\beta = a_1\alpha_1 + \cdots + a_{n-1}\alpha_{n-1}$ where $a_1, \ldots, a_{n-1}$ are all non-negative integers or all non-positive integers. Show that the angle between two different $\alpha_i, \alpha_j \in \mathbf{\Phi}$ is either $\pi/2$ or $2\pi/3$.

(e) Show that $<\alpha_i, \alpha_j> = 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i+1,j}$ for all $1 \leq i, j \leq n-1$.

(f) Compute the Cartan matrix with respect to the ordered basis Δ.

(g) Show that the Dynkin diagram of L is



where i represents α_i.

2. Suppose that $E = \boldsymbol{R}^2$ with the usual (positive definite) inner product and let $\boldsymbol{\Phi}$ be a rank two system of roots for E. *You may assume that* $(a, 0) \in \boldsymbol{\Phi}$ for some $a \in \boldsymbol{R}\backslash 0$, $\boldsymbol{\Phi} = \{\alpha_0, \ldots, \alpha_{2m-1}\}$ for some $m > 1$, and $u_i = \dfrac{\alpha_i}{||\alpha_i||}$ is given by

$$u_i = \begin{pmatrix} \cos\left(\frac{\pi}{m} i\right) \\ \\ \sin\left(\frac{\pi}{m} i\right) \end{pmatrix} \qquad (1)$$

for all $0 \leq i < 2m$. See §3.1.3 of the Class Notes.

For $i \in \boldsymbol{Z}$ we let u_i be defined by (1), we let $\tau_i = \tau_{u_i}$ be the reflection of E through the line $\boldsymbol{R}u_i$, and we let $\sigma_i = -\tau_i$ be the reflection of E through the hyperplane, or line in this case, $u_i^{\perp}$. (Note that $u_{m+i} = -u_i$, hence $u_{2m+i} = u_i$, for all $i \in \boldsymbol{Z}$. *You may assume that* $u_i = \pm u_j$ if and only if $j = \ell m + i$ for some $\ell \in \boldsymbol{Z}$.)

(a) Show that $\tau_i(u_j) = u_{2i-j}$ for all $i, j \in \boldsymbol{Z}$. [Hint: Since the u_i's have length 1 observe that $\tau_i(v) = 2(v, u_i)u_i - v$ for all $v \in E$. The calculation of $\tau_i(u_j)$ only involves some basic trigonometric formulas. Note that $\tau_{m+i} = \tau_i$, hence $\sigma_{m+i} = \sigma_i$, for all $i \in \boldsymbol{Z}$.]

(b) Show that $\sigma_i(u_j) = u_{m+2i-j}$ for all $i, j \in \boldsymbol{Z}$.

(c) Show that there are $a_i \in \boldsymbol{R}\backslash 0$ such that

  (1) $a_{i+2\ell} = a_i$ and
  (2) $a_{m+\ell} = a_\ell$ for all $i, \ell \in \boldsymbol{Z}$, and
  (3) $\alpha_i = a_i u_i$ for all $0 \leq i < 2m$.

  [Hint: Show that $\tau_i(\boldsymbol{\Phi}_{\mathsf{n}}) = \boldsymbol{\Phi}_{\mathsf{n}}$, where $\boldsymbol{\Phi}_{\mathsf{n}} = \{u_0, \ldots, u_{2m-1}\}$ and the listed elements are distinct.]

(d) Suppose $a_i \in \boldsymbol{R}\backslash 0$ for all $i \in \boldsymbol{Z}$ satisfies (1)–(2). Show that $\boldsymbol{\Phi}' = \{a_0 u_0, \ldots, a_{2m-1} u_{2m-1}\}$ satisfies axioms (R1)–(R3) of a root system for E.

3. Use the table on page 45 of the text together with Exercise 2 above to construct the rank 2 root systems, where $\alpha_0 = (1, 0)$. [Comment: If $\boldsymbol{\Phi} = \{\alpha_1, \ldots, \alpha_n\}$ is a root system and $a \in \boldsymbol{R}\backslash 0$, then $a\boldsymbol{\Phi} = \{a\alpha_1, \ldots, a\alpha_n\}$ is a root system. Thus we may "normalize" so that one of the roots has length 1.]

4. Let $\mathcal{W}_m$ be the subgroup of isometries of $E = \boldsymbol{R}^2$ generated by the reflections $\sigma_0, \ldots, \sigma_{2m-1}$ of Exercise 2 above. (This is the Weyl group of $\boldsymbol{\Phi}$.)

(a) Show that $\mathcal{W}_m$ is isomorphic to the subgroup W_m of $\mathrm{Sym}(\boldsymbol{\Phi}_{\mathsf{n}})$ generated by $\boldsymbol{\sigma}_0, \ldots, \boldsymbol{\sigma}_{2m-1}$, where $\boldsymbol{\sigma}_i(u_j) = u_{m+2i-j}$ for all $0 \leq i, j < 2m$. (Note: $u_{m+2i-j} = u_\ell$ where $0 \leq \ell < 2m$ and $m + 2i - j \equiv \ell \ (\mathrm{mod}\ 2m)$.

(b) Show that $\mathcal{W}_n \simeq D_{2m}$. [Hint: Note that $\tau = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_0$ has order m.]