MATH 531 **Written Homework 2 Solution** **Radford** **10/18/07**

1. (**20 points**) A is a Lie algebra over F, S is a subspace of L, $S^{(0)}$, $S^{(1)}$, $S^{(2)}$, ... defined inductively by $S^{(0)} = S$ and $S^{(i+1)} = [S^{(i)}\, S^{(i)}]$ for $i \geq 0$.

(a) (**4**) $(S^{(i)})^{(j)} = S^{(i+j)}$ for all $i, j \geq 0$; proof for fixed i by induction on j. $(S^{(i)})^{(0)} = S^{(i)} = S^{(i+0)}$. Thus the formula holds for $j = 0$.

Suppose $j \geq 0$ and the formula holds for j. Then $(S^{(i)})^{(j+1)} = [(S^{(i)})^{(j)} \quad (S^{(i)})^{(j)}] = [S^{(i+j)} \quad S^{(i+j)}] = S^{(i+j+1)}$. Thus $(S^{(i)})^{(j)} = S^{(i+j)}$ for all $i, j \geq 0$.

(b) (**4**) $S \subseteq T$, the latter a subspace of L. Then $S^{(i)} \subseteq T^{(i)}$ for all $i \geq 0$ by induction on i. The formula holds for $i = 0$ since $S^{(0)} = S \subseteq T = T^{(0)}$.

Suppose $i \geq 0$ and the formula holds. Then $S^{(i+1)} = [S^{(i)} \quad S^{(i)}] \subseteq [T^{(i)} \quad T^{(i)}] = T^{(i+1)}$. Thus $S^{(i)} \subseteq T^{(i)}$ for all $i \geq 0$.

(c) (**4**) $f : L \longrightarrow L'$ is a map of Lie algebras. Then $f(S^{(i)}) = f(S)^{(i)}$ for all $i \geq 0$ by induction on i. Since $f(S^{(0)}) = f(S) = f(S)^{(0)}$ the formula holds for $i = 0$.

Suppose $i \geq 0$ and the formula holds. Then $f(S^{(i+1)}) = f([S^{(i)} \quad S^{(i)}]) = [f(S^{(i)}) \quad f(S^{(i)})] = [f(S)^{(i)} \quad f(S)^{(i)}] = f(S)^{(i+1)}$. Thus $f(S^{(i)}) = f(S)^{(i)}$ for all $i \geq 0$.

(d) (**4**) Since S is an ideal of L, $S^{(0)} = S$ is an ideal of L. Suppose $i \geq 0$ and $S^{(i)}$ is an ideal of L. Since the product of ideals of L is an ideal of L, $S^{(i+1)} = [S^{(i)} \quad S^{(i)}]$ is an ideal of L. Thus $S^{(i)}$ is an ideal of L for all $i \geq 0$ by induction on i.

(e) (**4**) Since S is a subalgebra of L, $S^{(i)}$ is an ideal of S, hence a subalgebra of S, for all $i \geq 0$, by part (d).

2. (**20 points**) $L = \bigoplus_{i=0}^{\infty} L(i)$ is a graded Lie algebra over F, $L_i = L(i) \oplus L(i+1) \oplus L(i+2) \oplus \cdots$ for all $i \geq 0$. Since $[L(0) \quad L(0)] \subseteq L(0+0) = L(0)$ it follows that $L(0)$ is a Lie subalgebra of L. *We use an algebra of sets which would be good to justify in detail.*

(a) (**4**) $L_i = L(i) \oplus L_{i+1}$; thus $L_i \supseteq L_{i+1}$ for all $i \geq 0$ which means $L = L_0 \supseteq L_1 \supseteq L_2 \supset \cdots$ is a descending chain of subspaces of L. For $i, j \geq 0$ note

$$[L_i \quad L_j] = [\sum_{k=i}^{\infty} L(k) \quad \sum_{\ell=j}^{\infty} L(\ell)] = \sum_{k=i}^{\infty} \sum_{\ell=j}^{\infty} [L(k) \quad L(\ell)] \subseteq \sum_{k=i}^{\infty} \sum_{\ell=j}^{\infty} L(k+\ell) \subseteq \sum_{r \geq i+j} L(r) \subseteq L_{i+j}.$$

In particular $[L \quad L_i] = [L_0 \quad L_i] \subseteq L_{0+i} = L_i$ which means that L_i is an ideal of L.

(b) (**4**) $L(0)$ is abelian. $L^{(i)} \subseteq L_{2^{i}-1}$ for all $i \geq 1$ by induction on i. Since $L(0)$ is abelian and L_1 is an ideal of L, the calculation

$$\begin{aligned} L^{(1)} = [L \quad L] = [L(0) + L_1 \quad L(0) + L_1] &= [L(0) \quad L(0)] + [L(0) \quad L_1] + [L_1 \quad L(0)] + [L_1 \quad L_1] \\ &\subseteq [L(0) \quad L(0)] + L_1 = (0) + L_1 = L_{2^1-1} \end{aligned}$$

shows that the formula is true for $i = 1$.

Suppose $i \geq 1$ and the formula holds. Then $L^{(i+1)} = [L^{(i)} \;\; L^{(i)}] \subseteq [L_{2^{i-1}} \;\; L_{2^{i-1}}] \subseteq L_{2^{i-1}+2^{i-1}} = L_{2^i}$. Therefore the formula holds for all $i \geq 1$.

Suppose that $L(0)$ is abelian and $L(n) = L(n+1) = \cdots = (0)$ for some $n \geq 0$.

(c) (**4**) Then $L_n = L_{n+1} = \cdots = (0)$. As $2^n \geq n$ for all $n \geq 0$ we have $L^{(n+1)} \subseteq L_{2^n} = (0)$ by part (b). Thus L is solvable.

(d) (**4**) $K = [L \;\; L]$ is nilpotent. For $K^0 = K = L^{(1)} \subseteq L_1$ by part (b). Suppose $i \geq 0$ and $K^i \subseteq L_{i+1}$. Then $K^{i+1} = [K \;\; K^i] \subseteq [L_1 \;\; L_{i+1}] \subseteq L_{i+2}$. Thus $K^i \subseteq L_{i+1}$ or all $i \geq 0$. Consequently $K^n \subseteq L_{n+1} = (0)$ which means that $K = [L \;\; L]$ is nilpotent.

(e) (**4**) Suppose that $[L(0)\, L(1)] = L(1)$ and is not zero. Now $L^0 = L \supseteq L(1)$. Suppose that $i \geq 0$ and $L^i \supseteq L(1)$. Then $L^{i+1} = [L \;\; L^i] \supseteq [L(0) \;\; L(1)] = L(1)$. Therefore $L^i \supseteq L(1)$ for all $i \geq 0$ by induction on i. As $L(1) \neq (0)$, $L^i \neq (0)$ for all $i \geq 0$ which means that L is not nilpotent.

3. (**20 points**) First:

Let $n \geq 1$, $L = t(n, F)$, and let $L(i)$ be the span of the $e_{\ell\,\ell'}$'s, where $1 \leq \ell, \ell' \leq n$ and $\ell' = \ell + i$.

(a) (**7**) Since the $e_{i\,j}$'s, where $1 \leq i \leq j \leq n$, form a basis for L, and there is a partitioning of this basis whose cells form bases for distinct $L(i)$'s, $L = t(n, F) = \bigoplus_{i=0}^{\infty} L(i)$ is the direct sum of subspaces. Note that $L(i) \neq (0)$ if and only if $0 \leq i \leq n-1$.

Let $0 \leq i, j \leq n-1$ and consider typical basis elements $e_{\ell\,\ell+i}, e_{\ell'\,\ell'+j}$ for $L(i), L(j)$ respectively. Since $e_{\ell\,\ell+i} e_{\ell'\,\ell'+j} = \delta_{\ell+i\,\ell'} e_{\ell\,\ell'+j}$ it follows that this product is not zero if and only if $\ell + i = \ell'$, in which case $\ell' + j = \ell + (i+j)$ and $e_{\ell\,\ell+i} e_{\ell'\,\ell'+j} = e_{\ell\,\ell+(i+j)} \in L(i+j)$. *We have shown that with matrix multiplication $L(i)L(j) \subseteq L(i+j)$. Therefore the associative bracket $[L(i) \;\; L(j)] \subseteq L(i+j)$.*

Suppose $n > 1$. For $1 \leq i \leq n-1$ the calculation $[e_{i\,i} \;\; e_{i\,i+1}] = e_{i\,i} e_{i\,i+1} - e_{i\,i+1} e_{i\,i} = e_{i\,i+1}$ shows that $[L(0) \;\; L(1)] = L(1)$. Observe $L(1) \neq (0)$.

For distinct $1 \leq i, j \leq n$ the calculation $[e_{i\,i} \;\; e_{j\,j}] = e_{i\,i} e_{j\,j} - e_{i\,i} e_{j\,j} = 0$ shows that $L(0)$ is abelian.

When $n = 1$ we have $L(0) = Fe_{1\,1}$ is therefore abelian and $L(0) = (0)$; hence $[L(0) \;\; L(1)] = (0) = L(1)$.

(b) (**3**) Use Exercise 2 to conclude that L is solvable, $[L\, L]$ is nilpotent, and L is not nilpotent. **True only when** $n > 1$.

Now let L be the Lie algebra with basis $\{x, y\}$ determined by $[x\, y] = y$.

(c) (**7**) Set $L(0) = Fx$, $L(1) = Fy$, and $L(i) = (0)$ for all $1 < i$. Then $L = L(0) \oplus L(1) = L(0) \oplus L(1) \oplus L(2) \oplus \cdots$. Since $[L(0) \;\; L(0)] = (0) \subseteq L(0+0)$, $[L(1) \;\; L(0)] = [L(0) \;\; L(1)] = [Fx \;\; Fy] = Fy = L(1)$, and $[L(i) \;\; L(j)] = (0) \subseteq L(i+j)$ when $i + j > 1$, it follows that L has the structure of a graded Lie algebra and $L(0)$ is abelian.

(d) (**3**) By Exercise 2, L is solvable, $[L\, L]$ is nilpotent, and L is not nilpotent.

4. (**20 points**) Let L be a Lie algebra over F. Show that the following are equivalent:

(a) There exists a descending sequence of subalgebras

$$L = L_0 \supseteq L_1 \supseteq L_2 \supseteq \cdots \supseteq L_n = (0)$$

for some $n \geq 0$ such that L_{i+1} is an ideal of L_i and the quotient L_i/L_{i+1} is abelian for all $0 \leq i < n$.

(b) L is solvable.

(a) implies (b) (**12**) $L^{(0)} = L = L_0$. Suppose that $0 \leq i$ and $L^{(i)} \subseteq L_i$. Since L_i/L_{i+1} is abelian $[L^{(i)} \;\; L^{(i)}] + L_{i+1} = [L^{(i)} + L_{i+1} \;\; L^{(i)} + L_{i+1}] \subseteq [L_i + L_{i+1} \;\; L_i + L_{i+1}] = (\mathbf{0})$ shows $L^{(i+1)} = [L^{(i)} \;\; L^{(i)}] \subseteq L_{i+1}$. By induction $L^{(i)} \subseteq L_i$ for all $i \geq 0$. In particular $L^{(n)} \subseteq L_n = (0)$ and L is solvable.

(b) implies (a) (**8**) $L = L^{(0)} \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \cdots$ is a decreasing sequence of ideals and the quotient $L^{(i)}/L^{(i+1}$ is abelian as a result of the coset calculation

$$[L^{(i)} + L^{(i+1)} \;\; L^{(i)} + L^{(i+1)}] = [L^{(i)} \;\; L^{(i)}] + L^{(i+1)} = L^{(i+1)} + L^{(i+1)} = (0 + L^{(i+1)}) = (\mathbf{0}).$$

5. (**20 points**) Let L be a finite-dimensional nilpotent Lie algebra over F.

(a) (**10**) L is solvable and thus has a flag of ideals $(0) = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n = L$ by Corollary B, page 12 of the text.

Comment: We have just applied a 32-pound sledge hammer to solve this problem which needs only a delicate tap. Suppose that $L^i \supseteq L^{i+1}$ is a proper inclusion (which must be the case if $L^i \neq (0)$). Let V be a subspace of L such that $L^{i+1} \subseteq V \subseteq L^i$. The calculation $[L \;\; V] \subseteq [L \;\; L^i] = L^{i+1} \subseteq V$ shows that V is an ideal of L. Choose a basis $\mathcal{B}$ for L^{i+1} and let $\mathcal{B} \cup \{x_1, \ldots, x_s\}$ be an extension of $\mathcal{B}$ to a basis for L^i. (Note: $\mathcal{B} = \emptyset$ if $L^{i+1} = (0)$.) Let $m = \operatorname{Dim} L^{i+1}$ and define $L_{m+\ell} = \operatorname{span}(\mathcal{B} \cup \{x_1, \ldots, x_\ell\})$ for $1 \leq \ell \leq s$. (Thus $L_{m+s} = L^i$ and $m + s = \operatorname{Dim} L^i$.) Then $(0) = L_0, L_1, L_2, \ldots$ is the desired flag.

(b) (**10**) I is an ideal of L and the projection $\pi : L \longrightarrow L/I$ is a surjective Lie algebra map. Since $\mathcal{L} = L/I$ is nilpotent by part (a) of §3.2 Proposition. Thus there is a flag of ideals $(0) = \mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \cdots \subseteq \mathcal{L}_r = \mathcal{L}$ by part (a). Now $I_i = \pi^{-1}(\mathcal{L}_i)$ is an ideal of L for all $0 \leq i \leq r$ and $I = \operatorname{Ker} \pi = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_r = L$. Since π is surjective, the restriction $\pi|I_i : I_i \longrightarrow \pi(I_i) = \mathcal{L}_i$ is surjective. Thus by the Rank-Nullity Theorem

$$\operatorname{Dim} I_{i+1} = \operatorname{Dim} \operatorname{Im} \pi|I_{i+1} + \operatorname{Dim} \operatorname{Ker} \pi|I_i = \operatorname{Dim} \mathcal{L}_{i+1} + \operatorname{Dim} I = (\operatorname{Dim} \mathcal{L}_i + 1) + \operatorname{Dim} I = \operatorname{Dim} I_i + 1$$

for all $0 \leq i < r$.

Comment: Appealing to part (a), let $I'_i = L_i + I$ for all $0 \leq i \leq n$. Since the sum of ideals is an ideal, $I'_0 = L_0 + I = (0) + I = I \subseteq I'_1 \subseteq \cdots \subseteq I'_n = L_n + I = L + I = L$ is a chain of ideals of L. Let $0 \leq i < n$. Then $L_{i+1} = L_i \oplus Fv$ for some $v \in L$. Therefore $I'_{i+1} = I_i + Fv$ which means that $I'_{i+1} = I'_i$ or $\operatorname{Dim} I'_{i+1} = \operatorname{Dim} I'_i + 1$. Evidently the distinct terms of $I = I'_0, I'_1, \ldots, I'_n = L$ form the desired sequence.