We follow the notation of the text and that used in class. You may use results from the course materials on the class homepage and the text. This version replaces the previous one.

1. (35 points) $L_{\alpha_{k,\ell}} = R e_{k,\ell}$ and $L_{-\alpha_{k,\ell}} = L_{\alpha_{\ell,k}} = R e_{\ell,k}$.

(a) (5) Let $\alpha_{k,\ell} \in \Phi$. Then $k \neq \ell$. The calculation

$$e_{k,k} - e_{\ell,\ell} = [e_{k,\ell}, e_{\ell,k}] = \kappa(e_{k,\ell}, e_{\ell,k}) t_{k,\ell} = (2n \delta_{k,k} \delta_{\ell,\ell} - 2 \delta_{k,\ell} \delta_{\ell,k}) t_{k,\ell} = 2nt_{k,\ell}$$

establishes $t_{k,\ell} = \frac{1}{2n} (e_{k,k} - e_{\ell,\ell})$. See §8.3 Proposition (c) and WH4 Exercise 1(c).

(b) (5) Using part (a) observe that

$$\langle \alpha_{k,\ell}, \alpha_{r,s} \rangle = \alpha_{k,\ell}(b_{r,s}) = \frac{1}{2n} \delta_{r,s}$$

$$\langle \alpha_{k,\ell}, \alpha_{r,s} \rangle = \frac{1}{2n} \alpha_{k,\ell}(\sum_{i=1}^{n} (\delta_{i,r} - \delta_{i,s}) e_{i,i})$$

$$= \frac{1}{2n} (\delta_{k,r} - \delta_{k,s} - (\delta_{\ell,r} - \delta_{\ell,s}))$$

$$= \frac{1}{2n} (\delta_{k,r} + \delta_{\ell,s} - \delta_{k,s} - \delta_{\ell,r}) .$$

(c) (5) Using part (b) we note

$$\|\alpha_{k,\ell}\|^2 = \langle \alpha_{k,\ell}, \alpha_{k,\ell} \rangle = \frac{1}{2n} (\delta_{k,k} + \delta_{\ell,\ell} - \delta_{k,\ell} - \delta_{\ell,k}) = \frac{1}{n} .$$

Thus $|||\alpha_{k,\ell}||| = \sqrt{n}$. Thus

$$\cos \theta = \frac{\langle \alpha_{k,\ell}, \alpha_{r,s} \rangle}{|||\alpha_{k,\ell}||| |||\alpha_{r,s}|||} = \frac{1}{2n} (\delta_{k,r} + \delta_{\ell,s} - \delta_{k,s} - \delta_{\ell,r}) (\frac{1}{\sqrt{n}}) (\frac{1}{\sqrt{n}}) = \frac{1}{2} (\delta_{k,r} + \delta_{\ell,s} - \delta_{k,s} - \delta_{\ell,r}) .$$

(d) (5) Let $1 \leq i \leq j < k \leq n$ and $\sum_{u=1}^{n} \lambda_u e_{u,u} \in H$. Then

$$(\alpha_{i,j} + \alpha_{j,k})(\sum_{u=1}^{n} \lambda_u e_{u,u}) = (\lambda_i - \lambda_j) - (\lambda_j - \lambda_k) = \lambda_i - \lambda_k = \alpha_{i,k}(\sum_{u=1}^{n} \lambda_u e_{u,u})$$
which shows that $\alpha_{i,j} + \alpha_{j,k} = \alpha_{i,k}$. Now suppose that $1 \leq k < \ell \leq n$. By induction $\alpha_{k,\ell} = \sum_{i=k}^{\ell-1} \alpha_i$. Thus $\alpha_{\ell,k} = -\alpha_{k,\ell} = \sum_{i=k}^{\ell-1} -\alpha_i$.

Since $\Phi$ spans $H^*$ we have just shown that $\Delta = \{\alpha_1, \ldots, \alpha_{n-1}\}$ spans $H^*$. Since $\dim H^* = \dim H = n - 1$ it follows that $\Delta$ is a basis for $H^*$. Thus $\Delta$ is a base for $\Phi$.

Suppose $1 \leq i, j \leq n$ are distinct. By part (c)

$$\cos \theta = \frac{1}{2} (\delta_{i,j} + \delta_{i+1,j+1} - \delta_{i,j+1} - \delta_{i+1,j}) = \frac{1}{2} (\delta_{i,j+1} - \delta_{i+1,j})$$

which has the value 0 or $-\frac{1}{2}$. Thus angle between two different $\alpha_i, \alpha_j \in \Phi$ is either $\pi/2$ or $2\pi/3$.

(e) (5) We use the calculations of part (c) to deduce

$$\langle \alpha_i, \alpha_j \rangle = 2 \frac{(\alpha_{i,i+1}, \alpha_{j,j+1})}{\|\alpha_{i+1}\| \|\alpha_{j+1}\|} = 2 \frac{(\alpha_{i,i+1}, \alpha_{j,j+1})}{\|\alpha_{i+1}\| \|\alpha_{j+1}\|} = 2\left(\frac{1}{2} (\delta_{i,i} + \delta_{i+1,j+1} - \delta_{i,j+1} - \delta_{j,i+1})\right)$$

from which $\langle \alpha_i, \alpha_j \rangle = 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i+1,j}$ for all $1 \leq i, j \leq n - 1$ follows.

(f) (5) From part (e) observe that $\langle \alpha_i, \alpha_i \rangle = 2$, $\langle \alpha_i, \alpha_{i+1} \rangle = -1 = \langle \alpha_{i+1}, \alpha_i \rangle$ for all $1 \leq i < n - 1$ and $\langle \alpha_i, \alpha_j \rangle = 0$ otherwise. Thus the Cartan matrix is

$$
\begin{pmatrix}
2 & -1 & & & \\
-1 & 2 & -1 & & \\
& -1 & \ddots & \ddots & \\
& & -1 & 2 & \\
& & & & \\
\end{pmatrix}
$$

(g) (5) suppose that $1 \leq i, j \leq n$ are distinct. Then $\langle \alpha_i, \alpha_j \rangle = 0$ unless $j = i + 1$ or $i = j + 1$, in which case $\langle \alpha_i, \alpha_j \rangle \neq 0$, by part (f). Thus the Dynkin diagram of $L$ is

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•       •       •
1       2       n - 1
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where $i$ represents $\alpha_i$.

2. (25 points)

$$u_i = \begin{pmatrix}
\cos \left( \frac{\pi i}{m} \right) \\
\sin \left( \frac{\pi i}{m} \right)
\end{pmatrix}$$

for all $0 \leq i < 2m$. 

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(a) (7) \( \tau_i(u_j) = u_{2i-j} \) for all \( i, j \in \mathbb{Z} \) if and only if \( 2(u_j, u_i)u_i - u_j = u_{2i-j} \) for all \( i, j \in \mathbb{Z} \).

Fix \( i, j \in \mathbb{Z} \) and set \( a = \frac{\pi}{m}i \) and \( b = \frac{\pi}{m}j \). Part (a) comes down to establishing

\[
2 \begin{pmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{pmatrix} \begin{pmatrix} \cos a \\ \sin a \end{pmatrix} - \begin{pmatrix} \cos b \\ \sin b \end{pmatrix} = \begin{pmatrix} \cos(2a-b) \\ \sin(2a-b) \end{pmatrix},
\]

or equivalently

\[
2(\cos a \cos b + \sin a \sin b) \begin{pmatrix} \cos a \\ \sin a \end{pmatrix} - \begin{pmatrix} \cos b \\ \sin b \end{pmatrix} = \begin{pmatrix} \cos(2a-b) \\ \sin(2a-b) \end{pmatrix},
\]

or equivalently

\[
2 \cos(a-b) \cos a - \cos b = \cos(2a-b)
\]

and

\[
2 \cos(a-b) \sin a - \sin b = \sin(2a-b).
\]

As for (2) observe that

\[
\cos(2a-b) = \cos((a-b) + a)
\]

\[
= \cos(a-b) \cos a - \sin(a-b) \sin a
\]

\[
= \cos(a-b) \cos a - (\sin a \cos b - \cos a \sin b) \sin a
\]

\[
= \cos(a-b) \cos a - (1 - \cos^2 a) \cos b + \cos a \sin b \sin a
\]

\[
= \cos(a-b) \cos a - \cos b + (\cos a \cos b + \sin b \sin a) \cos a
\]

\[
= 2 \cos(a-b) \cos a - \cos b.
\]

Replacing \( a \) with \( \pi/2 - a \) and \( b \) with \( \pi/2 - b \) in (2) gives (3).

(b) (5) \( \sigma_i(u_j) = -\tau_i(u_j) = -u_{2i-j} = u_{m+2i-j} \).

(c) (8) Let \( a_i = ||a_i|| \) for all \( 0 \leq i < 2m \). Then \( a_i = a_iu_i \) for all \( 0 \leq i < 2m \) and (3) is satisfied. We extend the definition of \( a_i \) to all \( i \in \mathbb{Z} \) by passing from representatives of the cosets of \( 2m\mathbb{Z} \) in \( \mathbb{Z} \) by setting \( a_i+2m\ell = a_i \) for all \( \ell \in \mathbb{Z} \). Set \( a_i = a_iu_i \) for all \( i \in \mathbb{Z} \). Then the \( a_i \)'s constitute \( \Phi_n \) and \( a_{i+2m} = a_i \) for all \( i \in \mathbb{Z} \).

Note that \( \sigma_i \) is length preserving. Thus \( \sigma_i(u_j) = \frac{1}{||a_j||} \sigma_i(a_j) = \frac{1}{||\sigma_i(a_j)||} \sigma_i(a_j) \in \Phi_n \)

for all \( i, j \in \mathbb{Z} \) which shows that \( \sigma_i(\Phi_n) = \Phi_n \) for all \( i \in \mathbb{Z} \).

To show (1), suppose \( i, j \in \mathbb{Z} \). Then \( a_ju_{m+2i-j} = \sigma_i(a_j) \in \Phi \). Therefore \( \sigma_i(a_j) \) and \( \sigma_{m+2i-j} \) are scalar multiples which means that \( a_ju_{m+2i-j} = \pm a_{m+2i-j}u_{m+2i-j} \). Since the \( a_i \)'s are positive, \( a_j = a_{m+2i-j} \), or equivalently \( a_j = a_{m+2i-j} \) for all \( i, j \in \mathbb{Z} \). This establishes (1) and (2).

(d) (5) Let \( \alpha'_i = a_iu_i \) for all \( i \in \mathbb{Z} \), where the \( a_i \)'s satisfy (1)–(2) above, and let \( \Phi' \) be the set of the \( \alpha'_i \)'s. Note that \( \alpha'_i = \alpha'_{i+2m} \) for all \( i \in \mathbb{Z} \). Since \( \Phi' \) spans \( E \), so does \( \Phi_n \) and therefore so does \( \Phi' \). Since \( 0 \notin \Phi_n \) and the \( a_i \)'s are not zero, \( 0 \notin \Phi' \). Thus (R1) is satisfied.
(R2). Let \( i, j \in \mathbb{Z} \). Then 
\[ -\alpha_i' = a_i(-u_i) = a_{m+i}u_{m+i} \in \Phi'. \]

Suppose that \( \alpha_i' = c\alpha_j' \) for some \( c \in \mathbb{R} \). Then \( u_i \) and \( u_j \) are scalar multiples. Therefore \( i = j + km \) for some \( k \in \mathbb{Z} \). If \( k \) is even then \( u_i = u_j \) and if \( k \) is odd then \( u_i = -u_j \). Now \( a_i = a_j \) in either case. Therefore \( \alpha_i' = \pm \alpha_j' \).

(R3). Let \( i, j \in \mathbb{Z} \). First of all note that \( \sigma_i = \sigma_{\alpha_i} \). Now \( \sigma_i(\alpha_j') = a_ju_{m+2i-j} = a_{m+2(i-j)+j}u_{m+2i-j} = a_{m+2i-j}u_{m+2i-j} \in \Phi'. \]

3. (20 points) In light of Exercise 3 and the table on page 45 of Humphreys \( m = 2, 3, 4, \) or 6 and, since \( \alpha_0 = u_0 \), the problem is to find a positive \( b \in \mathbb{R} \) (which will be the length of \( \alpha_1 \)) such that with
\[
a_{2\ell} = 1 \quad \text{and} \quad a_{2\ell+1} = b
\]for all \( \ell \in \mathbb{Z} \) the conditions
\[
a_{m+\ell} = a_\ell \quad \text{and} \quad 2 \left( \frac{a_j}{a_i} \right) (u_i, u_j) = <a_ju_j, a_iu_i> \in \mathbb{Z}
\]for all \( \ell, i, j \in \mathbb{Z} \). Without loss of generality we may assume \( b \geq 1 \) (that is \( \alpha_0 \) is a root of minimal length).
\[
(u_i, u_j) = \cos \left( \frac{\pi}{m}(i-j) \right)
\]
for all \( i, j \in \mathbb{Z} \). Suppose (4) holds. Observe that \( a_{m+\ell} = a_\ell \) for all \( \ell \in \mathbb{Z} \) holds automatically when \( m \) is even and holds if and only if \( b = 1 \) when \( m = 3 \). By considering whether or not \( i-j \) is even or odd we see that (5) is equivalent to
\[
b = 1 \text{ if } m = 3 \text{ and } 2 \cos \left( \frac{\pi}{m}(2\ell) \right), 2b \cos \left( \frac{\pi}{m}(2\ell+1) \right), 2b^{-1} \cos \left( \frac{\pi}{m}(2\ell+1) \right) \in \mathbb{Z}
\]for all \( \ell \in \mathbb{Z} \).

Case 1: \( m = 2 \). Here \( \cos \left( \frac{\pi}{m}(2\ell) \right) = \cos(\pi \ell) \in \{-1, 1\} \) and \( \cos \left( \frac{\pi}{m}(2\ell+1) \right) = 0 \) for all \( \ell \in \mathbb{Z} \). Any \( b \geq 1 \) works.

Case 2: \( m = 3 \). Since \( \cos \left( \frac{n\pi}{3} \right) \in \{-\frac{1}{2}, 0, \frac{1}{2}\} \) for all \( n \in \mathbb{Z} \) condition (6) is met.

Case 3: \( m = 4 \). Here \( \cos \left( \frac{\pi}{m}(2\ell) \right) = \cos \left( \frac{\pi}{2}(\ell) \right) \in \{-1, 0, 1\} \) for all \( \ell \in \mathbb{Z} \). Since \( \cos \left( \frac{\pi}{m}(2\ell+1) \right) = \cos \left( \frac{\pi}{4}(2\ell+1) \right) \in \{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\} \), the remainder of (6) is satisfied if and only if \( \sqrt{2}b, \sqrt{2}b^{-1} \in \mathbb{Z} \). Writing \( b = \frac{m}{\sqrt{2}}, b^{-1} = \frac{n}{\sqrt{2}} \) for some \( m, n \in \mathbb{Z} \), it is easy to see that the remainder of (6) holds if and only if \( b = \sqrt{2} \).

Case 4: \( m = 6 \). Here \( \cos \left( \frac{\pi}{6}(2\ell) \right) = \cos \left( \frac{\pi\ell}{3} \right) \in \{-\frac{1}{2}, 0, \frac{1}{2}\} \) for all \( \ell \in \mathbb{Z} \). Since
\[
\cos \left( \frac{\pi}{m}(2\ell+1) \right) = \cos \left( \frac{\pi}{6}(2\ell+1) \right) \in \{-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\},
\]
the remainder of (6) is satisfied if and only if $\sqrt{3}b, \sqrt{3}b^{-1} \in \mathbb{Z}$. Mimicking the argument in the last part of Case 3 we see that the remainder of (6) holds if and only if $b = \sqrt{3}$.

Note: The diagrams on page 44 of Humhreys are exactly those which arise in the cases above.

4. (20 points) (a) Given $i \in \mathbb{Z}$. Since $\sigma_i$ is an isometry of $E$ and $\sigma_i(\Phi) = \Phi$, it follows that $\sigma_i(\Phi_n) = \Phi_n$. Therefore $W_m$ is a subgroup of $G_{\Phi_n}$. Now $\Phi_n$ spans $E$ since $\Phi$ does. Thus the restriction map $G_{\Phi_n} \rightarrow \text{Sym}(\Phi_n)$ given by $\sigma \mapsto \sigma$ is an injective group homomorphism. Note $\sigma_i(u_j) = u_{m+2i-j}$ for all $j \in \mathbb{Z}$ by part (b) of Exercise 2.

(b) Let $\tau = \sigma_1 \sigma_0$. The calculation $\tau(u_j) = \sigma_1(\sigma_0(u_j)) = \sigma_1(u_{m-j}) = u_{m+2-(m-j)} = u_{2+j}$ shows that $\tau(u_j) = u_{2+j}$ for all $j \in \mathbb{Z}$. Thus by induction $\tau^\ell(u_j) = u_{2\ell+j}$ for all $0 \leq \ell$ and $j \in \mathbb{Z}$. Since $u_k = u_\ell$ if and only if $k \equiv \ell \pmod{2m}$, we conclude that $\tau^\ell = \text{Id}_{\Phi_n}$ if and only if $m|\ell$. Therefore $\tau$ has order $m$. For $0 \leq \ell$ the calculation

$$\tau^\ell \sigma_0(u_j) = \tau^\ell(\sigma_0(u_j)) = \tau^\ell(u_{m-j}) = u_{2\ell+m-j} = \sigma_\ell(u_j)$$

for all $j \in \mathbb{Z}$ shows that $\sigma_\ell = \tau^\ell \sigma_0$. In particular $W_m$ is generated by $\tau$ and $\sigma_0$.

Let $N = \langle \tau \rangle$ and $H = \langle \sigma_0 \rangle$. Since $\tau$ has order 2 for all $i \in \mathbb{Z}$ we conclude that $\tau^\ell \sigma_0 = (\tau^\ell \sigma_0)^{-1} = \sigma_0^{-1} \tau^{-\ell} = \sigma_0 \tau^{-\ell}$, thus

$$\tau^\ell \sigma_0 = \sigma_0 \tau^{-\ell}, \quad (7)$$

for all $0 \leq \ell$. Therefore $NH = HN$ which means $HN$ is a subgroup of $W_m$. Since $N$ and $H$ generate $W_m$, $W_m = HN$ and $N$ is a normal subgroup of $W_m$.

Note that $\sigma_0 \in N$ if and only if $\sigma_0 = \tau^\ell$ for some $0 \leq \ell$ if and only if $u_{m-j} = u_{2\ell+j}$ for all $j \in \mathbb{Z}$ if and only if $m - 2(j + \ell) \equiv 0 \pmod{2m}$ for all $j \in \mathbb{Z}$. The latter implies $m \equiv 0 \pmod{2m}$, a contradiction. Therefore $\sigma_0 \notin N$.

We have shown that $|W_m| = 2m$. From (7) we conclude $\sigma_0^{-1} \tau \sigma_0 = \tau^{-1}$. Since $\sigma_0, \tau$ generate $W_m$ and $\sigma_0^2 = \text{Id}_{\Phi_n} = \tau^m$, $W_m \simeq D_{2m}$. 

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